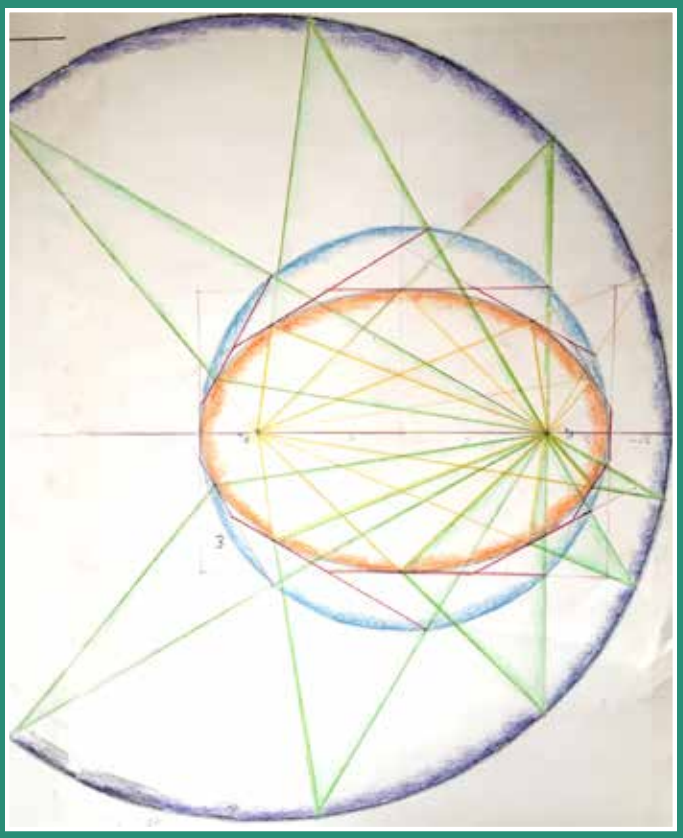
A Waldorf High School Mathematics Program

RATIONALE AND SOME SUGGESTIONS

Based on the Insights of Amos Franscescelli in Connection with Student Developmental Stages and Interests



Robert C. Oelhaf

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A Waldorf High School Mathematics Program with Appendices

RATIONALE AND SOME SUGGESTIONS

A teacher's course notes by R.C. Oelhaf
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Introduction

Practical life today requires facility with science and mathematics, as well as more directly technical matters such as engine mechanics and computer science. For the latter is built up on the foundation of the former. A full high school mathematics program is required for understanding the modern world. Furthermore mathematics is our introduction to pure (sense-free) thinking. And it is a major part of college entrance examinations.

The program is naturally first of all based on Rudolf Steiner's indications (See especially E.A. Karl Stockmeyer, *Rudolf Steiner's Curriculum for Waldorf Schools*). However the original Waldorf school was burdened by the need to prepare students for the onerous state graduation examinations, as well as rigorous expectations at earlier grade levels. For example, logarithms were an important part of the school leaving exams.

Today this is ancient history. No one uses logarithms for calculation; slide rules are exhibits in museums. Waldorf school traditions also offer guidance. A mathematics program needs to respond to the developmental stages of high school students and their interests at each grade level. The program also needs to support the science program and include an introduction to computer structure and function. And it needs to prepare students for the SAT and for further study of mathematics and science in college.

In setting up the program at Hawthorne Valley High School, we were indeed fortunate to be able to work closely with long-time Waldorf mathematics teacher, Amos Francescelli of the Rudolf Steiner School in New York, for the overall structure of the program, for details of particular blocks and for an approach to teaching individual students and classes. We also practiced peer- and self-evaluation through regular reports to high school faculty meetings, a form which is highly recommended. Each teacher was expected to explain what and why for each course or block. This is standard form for the top educational systems in the world today, e.g., Finland's.

A High School Mathematics Program

There are two major branches of introductory mathematics, which Amos used to call the "two eyes" of mathematics, namely Geometry and Algebra. Generally each year the work in each branch is carried forward in its own way until they merge in 11th grade into Analytic Geometry.

9th Grade: Geometry block: Loci, including Conic Sections, Engineering Drawing
Algebra block: Probability and Statistics
Track class: Algebra I

10th Grade	Geometry block: Surveying
	Algebra block: Logic, Number bases, Golden section, Series
	Track class: Plane Geometry
11th Grade	Geometry block: Projective Geometry
	Algebra block: Analytic Geometry, Spherical Trigonometry
	Track class: Algebra II
12th Grade	Geometry and Algebra block: Current Mathematical Issues
	Track class: Calculus

Because of the pressure from other subjects, it is not always possible to have all the mathematics blocks one might desire. We must be ready to spring in when the opportunity arises and also be able to squeeze potential block material into track classes when necessary.

The SAT

Waldorf schools can prepare students for the SATs as a matter of course, through the normal curriculum, with some adjustments and emphases to address particular question types. Students who learn what they meet in the track classes and blocks should be prepared for the math sections of the SATs if they in addition receive guidance on how to take tests, practice some tests in class and work on practice tests on their own.

The SATs contain primarily questions in the areas of Algebra and Geometry. In addition there are questions requiring basic probability theory and also series. Waldorf schools can provide sufficient training in the regular curriculum when fully implemented. Since individual SAT questions generally are posed in English, rather than simple mathematics, special emphasis must be placed on verbal communication of assignment questions. Relevant questions from SAT tests would be good choices, in order to become accustomed to the SAT style of posing questions.

9th Grade track class is Algebra I.
9th Grade Algebra block is Probability and Statistics.
10th Grade track class is Geometry.
10th Grade Algebra block includes Series.

Students who study and learn the content of these courses, with content adjusted to meet SAT expectations, should be able to handle the SAT mathematics questions. In addition, students take practice tests in the beginning of 11th grade and are encouraged to continue taking practice tests on their own. Serious students will do this. Questions they meet which are not readily comprehensible should be brought to math class for clarification.

Extensive SAT practice during regular mathematics classes is not a good use of time. Some students will not be taking the SAT tests. Many excellent colleges and universities no longer require the SAT. For example, American University waives the SAT for students who apply early in the Fall. Others do not care at all or refuse to look at the scores.

Waldorf students make such an impressive application through their Portfolios and their personal interviews that the SAT score's importance is significantly diminished. (However some university systems, e.g., the New York State university system, do take SAT scores seriously.) For more SAT practice in school, it can be an option during electives.

Waldorf schools provide their students with the best possible foundation for learning math, namely the general curriculum—which includes much music, art, handwork, and movement through eurythmy and gymnastics—and personal contact with teachers, the factors which build brain cells. This fact should be communicated to parents. (See "A Case for Waldorf Education," R.C. Oelhaf, *Renewal,* November 2013)

Textbooks?

There are fine Waldorf high school programs, e.g., Jamie York's, and of course conventional mathematics programs. The McDougal Littell series of textbooks is student-friendly and covers all the bases. There are clear explanations and three levels of questions after each section. I did not find their attempt at posing SAT questions useful, however. Questions can easily be gleaned from actual SATs. This sequence, in grades 9, 10 and 11 is a good fit for a Waldorf school and for the SAT, as well as preparation for more advanced math for those so inclined.

Naturally each teacher will develop his or her own approach to the blocks and track classes to be taught. Making use of a textbook introduces flexibility for teacher and student. Teachers are of course free to supplement or replace material. And students have a convenient reference in cases of absence or difficulty in understanding the classroom presentation. Although it may be an ideal for teachers to prepare and distribute their own problem sets, this is not only an extremely demanding task, but makes it difficult for students who have missed work, e.g., due to illness, travel, foreign exchange, to catch up. Teachers should be creative with their own presentations in the classroom. Unless they have unlimited time or are writing a textbook themselves, they can well let the published textbook do the routine work.

In the Classroom

No matter what the subject, it is not always easy to elicit student interest and attention, particularly at the start of the lesson. And mathematics is not every student's "cup of tea." When a particular theorem or topic can be enlivened with stories of the individuals pioneering these areas, the general student can find a connection which would be missing from a strictly mathematical presentation, no matter how clear. Practical applications show the importance of the material for the world around us. An artistic approach involving color and drawings can be stimulating and help draw students in, addressing emotions as well as thinking. Challenging thought questions can wake students up to the mathematics class. And the consciousness of the teacher is also crucial.

It was remarkable how Amos Franscescelli, as a visiting teacher, could begin a class on time and with the direct engagement of the students, whether well-known or new to him. There were two things clear to students immediately: Amos was totally engaged with and interested in the mathematics topic of the day, and he was equally interested in each student in the classroom. He seemed fascinated by the mathematical theorems or problems on the agenda, as if he had never seen them before. As he continued the lesson, he seemed to be developing the laws and solving the problems for the first time, along with the students. He was completely engaged with seeing how each student also progressed in his or her understanding, seemed really interested, even fascinated.

This makes sense. After all, mathematics is a matter of pure thinking and can be developed by anyone with the skill and knowledge out of his or her own thought. Students recognize the interest of the teacher, they sense it, they know that what they think and grasp is important to the teacher. Student attention was not a problem. Neither was beginning the lesson. Amos always had some question from the news, some practical application of the mathematical topic before them, a thought problem or enigma for reflection or solution. Or he had some challenging mathematical exercise or problem. Sometimes these questions carried over from one day to the next, which also built in some suspense.

Where did this come from? Certainly Amos is first of all a mathematician and an earnest one. In fact his whole demeanor is earnest. He is not a joke teller. Amos is also a committed anthroposophist, and he underststands and appreciates the struggles of teenagers. (Amos had two grown sons.) I have no doubt that the teachers' meditation was a key factor, the calling to mind of each student in one's classes the evening before, which often leads directly to some insight into the students and the coming morning's lessons.

Grade 9 Math

Students entering the 9th Grade are coming strongly into themselves, feeling the exuberant energy of puberty, often quite sure of their judgments, which are often strongly held, as clear to them as black and white. And they are experiencing such inner energy as to almost border on chaos, uncertain if they can control themselves and their environment, much of which may be in some ways new. They want to understand themselves and the larger world around them. They need confidence in their ability to meet the uncertainties and to create from the energy and occasional chaos some order and even beauty.

The mathematics program in the 9th Grade seeks to address the students where they are at this stage, through the two 9th Grade blocks: Loci is the geometry block and Probability and Statistics is the algebra block for the 9th Grade. The track class has been traditionally Algebra I, which is in many ways perfectly appropriate for this level.

Algebra or Geometry?

There is currently a controversy among American mathematics teachers, approximately half wishing to hold to the traditional Algebra I, the others arguing that pushing Algebra I down into 8th grade and placing Geometry in 9th would allow more advanced math in the upper grades. While the change is possible, it is really not appropriate, considering the students' general development in this year. Plane geometry, with its logic and proofs, is right for 10th graders, ready for logical thinking and using this thinking for understanding and working with the real world. 9th graders are in general more interested in hands-on experience than in approaching the world through logical thinking.

If students have had an intensive algebra program in 8th grade, then Algebra should be taught anyway in 9th and called "Intermediate Algebra." Material which could not fit into the blocks could also be added to the track class, for example, more mechanical drawing, something Rudolf Steiner often recommended, and also an introduction to projective drawing in preparation for projective geometry.

9th Geometry Block: Loci, or Movement under Law

In this block students experience how following rules or laws produces practical solutions and beautiful forms. If all goes right, they will get the message without any moralizing or explicit statement.

Most students have learned the basic constructions of perpendiculars, bisectors, triangle centers. These need to be briefly reviewed, perhaps extended to three dimensions, now viewed as locations (loci) or paths governed by certain laws.

There are two ways to specify a locus: a static definition and a dynamic definition. The static definition describes the location of points which satisfy a condition, such as being equidistant from two points. The dynamic definition describes the movement of a point which fulfills the condition. For the perpendicular bisector we would have:

> **Static Definition:** The locus of all points in a plane equidistant from two fixed points is the perpendicular bisector of the line joining the two points.
>
> **Dynamic Definition:** The locus of a point moving in a plane such that it remains equidistant from two fixed points is the perpendicular bisector of the line joining the two points.

Students should be encouraged to imagine a creative name for their drawings, such as "Scylla and Charybdis" for the equal distances from two points. (What would unequal distances, such as a ratio of 1:2, look like?)

Some practical applications can be fun and a change of pace from the series of drawings. An introduction to some projective geometry concepts arises through the four incircles of the complete triangle. (Some introduction to projections would be appropriate if time permits. For exercises, see Bernhardt, *Projective Geometrie*. Although the text is in German, exercises may be copied.) Then the students should move on to the complexity of the nine- point circle, discovering a few more of the many secrets hidden in a triangle.

Next students explore ways of drawing the conics. Perhaps they have drawn parabolas or ellipses in the Lower School. They can learn other ways to do it, in particular as envelopes, the meaning of the names, the generating line or, in general, circle. Practical applications should be discussed for each form, also uses in three dimensions (football, submarine, reflector, cooling tower, saddle...) They can also be sketched. A list of suggested drawings is in the Appendix with some examples and supporting material.

Drawings should be done in pen and black ink, on high quality paper, without mistakes or erasures, with numbers, titles and borders. The work should be done in black and white. A good drawing pen such as a Micron is necessary, not a ball-point, which can leave small blots and smudges.

The loci drawing may culminate with the Cassini ovals, the locus of all points whose product of distances from two fixed points is constant. These curves include a variety

of forms, one of which is the lemniscate. Surprisingly, the form splits into two parts as it shrinks. A convenient set of radii is found in the Appendix. The main forms may be given names such as The Two Eyes, The Lemniscate, The Peanut, The Race Track, Round Table or Oval. Other possible complex drawings include the cycloid, epicycloid, strophoid, sissoid.

The drawing of the conic sections is an important foundation for later work in analytic geometry in the 11th Grade. Although a cone is simple enough, most three-dimensional conics are difficult to draw. An exception is a hyperboloid of one sheet. An example is shown in the Appendix. Also a saddle shape is possible, the hyperbolic paraboloid. All the three-dimensional forms are illustrated in Larson, et al., *Calculus* section 12.6, pp. 756f (4th ed) together with their equations.

Steiner urged taking technical drawing as far as the intersection of a cylinder and cone. By omitting some of the simple drawings, time would be freed up for some engineering drawing. Just as we learn to do long division, even though everyone uses calculators nowadays, some engineering drawing should be done by hand as well, although professionals use Computer Aided Design (CAD). Students should experience the difference between perspective, isometric and orthogonal. Some examples are shown in the Appendix. The drawing of cylinder and cone intersection is challenging, especially without drawing instruments (Appendix).

9th Algebra Block: Probability and Statistics

Some important elements of this block are: Counting. Permutations and Combinations. factorials.

Pascal's Triangle. diagonals, summation, Binomial Theorem, "The 12 Days of Christmas"

Note that the first diagonal is all ones, the second the natural numbers, the third the sum of the first natural numbers (e.g., the 10th, namely 55, the sum of the first 10 natural numbers is the total score in a cross country match between two teams). The fourth diagonal is the double sum, namely the sum of the sums up to that number. Example: The total number of presents in the song, "The Twelve Days of Christmas" is $n(n+1)(n+2)/2x3 = 364$, that is, the number of days in the year (minus one). This is the twelfth entry in the third row. Set the challenge to discover the total number. The meaning is that the 12 holy nights have within them the whole rest of the year. The birds are all English field birds.

Before leaving the triangle, tell the story of the young Karl Friedrich Gauss in math class, when the teacher told the class to add all the numbers from one to one hundred, then settled back for a relaxing hour. Before he could do anything, however, before him stood

Gauss, placing his tablet with his answer face down on the teacher's desk, as was the custom. Severely angry at Gauss for not doing the assignment, the teacher waited until all had placed their slates on top of the stack. Then he read off the answers from top to bottom. Gauss was the only one to get the right answer! He had apparently immediately grasped the rule, perhaps by adding, e.g., 50 to 51, 49 to 52, etc.

Probability. Choice, combinations, permutations...The SAT generally has questions involving probability. Typical SAT questions can be explained and practiced.

Statistics: collecting and presenting information, reading graphs, charts, etc.

Reference: Charles D. Miller, et al., *Mathematical Ideas*. This is a popular college-level textbook with many editions available used. Some, e.g., the 9th edition, have an interesting chapter on the Impossibility of Voting.

9th Track Class: Algebra I

Algebra is absolutely fundamental for most areas of technology in the modern world and for any further work in mathematics. It is also presupposed for the 10th Grade Physics block, Mechanics.

Students having trouble with the order of operations may want to remember the mnemonic device, "Please excuse my dear Aunt Sally," meaning in translation: "parentheses exponents multiplication division addition subtraction."

The SAT Mathematics sections are to a large extent basic algebra. Particularly important for solving SAT questions simply and efficiently is facility with ratio and proportion. Thus this chapter needs to be emphasized and practiced until it is second nature. All other basic algebraic operations must be thoroughly mastered, especially working with signed numbers. Functions is also a very important chapter, turning up on the SAT and also in advanced work in high school.

The quadratic formula is not easy for some students to remember. However many enjoy the singing the theorem to the tune of "Pop Goes the Weasel":

X equals negative b
Plus or minus the square root
Of b squared minus 4* a c
All over 2 a.

Notes: *The word "four" needs to be spread out over two notes, that is, spoken with two syllables. I once had a young lady in class, a dancer with considerable artistic talent and no connection to mathematics. She loved the song and sang it at every opportunity.

Computer Science

Students should be able to open up a modern computer and identify and understand the elements and their functions and relations. If input is still primarily through keyboard rather than voice, students need to learn touch typing. This does not require a large commitment of time. Only an introduction is necessary: how to use the teaching materials and enough practice so that they see how to continue on their own. Free lessons are available online. Practice in class may continue as time permits, for example, in an elective. Touch typing is so much more efficient than any other method of keyboard input, that it certainly is worthwhile to learn. It will be useful all through high school for essays and reports, and certainly for any college or university work.

9th Grade Appendices

9-1 Loci: List of Plates
9-2 Buried Treasure Problems
9-2 City Planning Problems
9-4 The Nine-point Circle
9-5 Proof of Tangent Line Construction for a Parabola
9-6 Conic Sections: Sections of a Cone
9-7 Hyperboloid of One Sheet
9-8 Drawing Cassini Curves
9-9 Chonchoid of Nicomedes
9-10 Isometric Drawing of Beveled Block
9-11 Orthographic Projection of Beveled Block
9-12 Intersection of Cone and Cylinder
9-13 Pythagorean Theorem: a Proof
9-14 Powers of Binomials
9-15 Families of Parabolas
9-16 Quadratic Equation Solved Three Ways
9-17 Theorem: Two Perpendicular Lines and Their Slopes

9-1

Loci: List of Plates

IN A PLANE:

1. Dog tied to a Post
Static Definition: The locus of all points equidistant from a point is a circle.
Dynamic Definition: The locus of a point moving so as to keep an equal distance from a fixed point is a circle.

2. Dog on a Run: The locus of all points equidistant from a line is two lines parallel to the original line.

3. Scylla and Charybdis: The locus of all points equidistant from two points is the perpendicular bisector of the line joining the points.

4. Two Highways: The locus of all points equidistant from two intersecting lines is two perpendicular lines bisecting the angles formed by the lines.

5. Circumcenter: The locus of points equidistant from three points is the circumcenter of the triangle formed by lines joining the three points.

6. Incenter: The locus of points equidistant from the sides of a triangle is the incenter.

7. Four Incenters: Extending the sides of a triangle, a total of four triangles is formed, each with its own incenter.

8. The Orthocenter: The three altitudes of a triangle meet in the orthocenter.

9. The Centroid: The three medians of a triangle meet in the centroid, the center of mass.

10. The Nine-point Circle: (Note: This plate will need an extra large sheet.)

11. Buried Treasure Problems

12. City Planning Problems

13. The Parabola as a locus of all points equidistant from a point (the Focus_ and a line

(the Base). Light from the focus is reflected parallel to the central axis.

14. The Parabola as the envelope of its tangents

15. The locus of all points the sum of whose distances from two fixed points is constant is an **Ellipse**. Light from one focus is reflected through the other.

16. The locus of all points the difference of whose distances from two fixed points is constant is a **Hyperbola.**

17. Three Conic Sections
Ellipse (*Ellipsis* means something missing.) Eccentricity < 1. Here $\varepsilon = \frac{1}{2}$.
Parabola (*Parable* means a likeness.) Eccentricity = 1
Hyperbola (*Hyperbole* means exaggeration.) Eccentricity > 1. Here $\varepsilon = 2$.

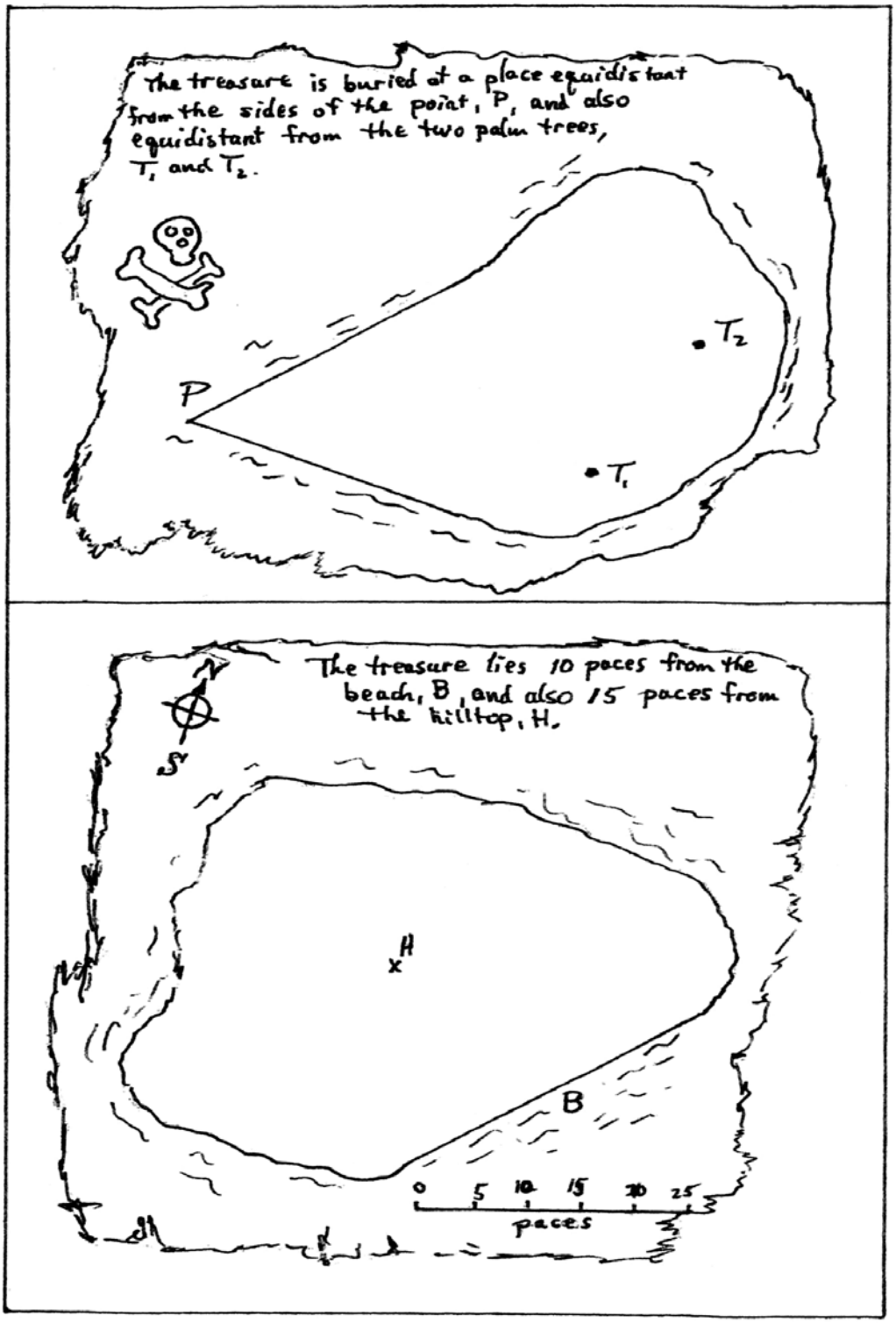
18. Conic Sections as Slices of a Cone, Conic Sections in 3D
Soccer Ball (Sphere)
Football/Submarine/Earth (Paraboloid)
Saddle (Hyperbolic Paraboloid)
Hyperboloid of One Sheet (Cooling tower)
Trough, Coaxial Cable, Doughnut/Torus

19. Cassini Curves: Two Eyes, Lemniscate, Peanut, Racetrack, Roundtable

OTHER POSSIBILITIES:
Perspective, Isometric and Orthogonal Drawings

9-2

Buried Treasure Problem



City Planning Problems

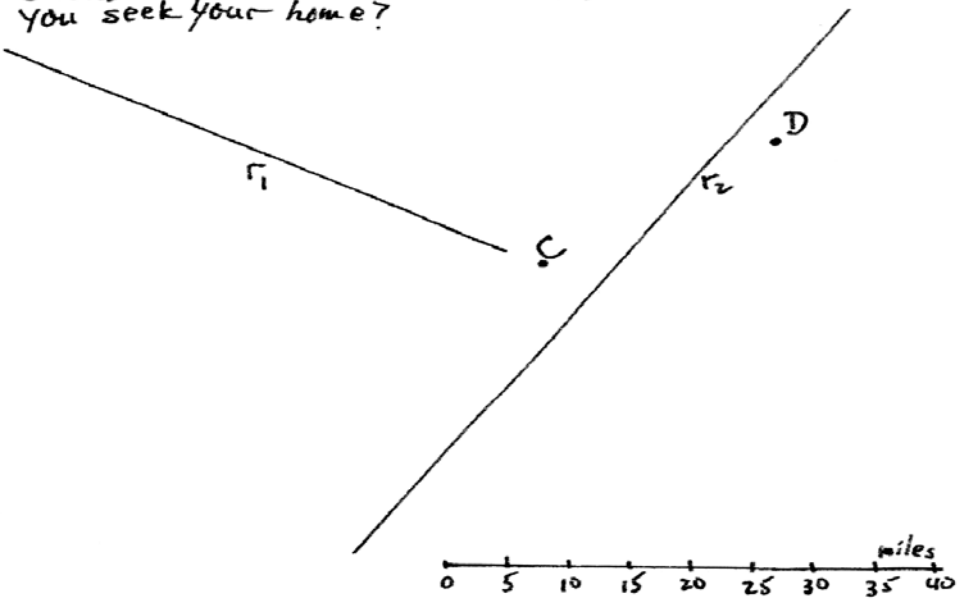
1. A factory is to be built at equal distances from three cities. What should its location be?

Albany

Boston

New York

2. You want to live at least 5 miles from the city center, C, but not farther away than 15 miles. Also you wish to be at least 20 miles from the dump, D. And you want to be at least 5 miles away from the main roads, r_1 and r_2. Where should you seek your home?



9-4

The Nine-point Circle

Sometimes called **Euler's Circle**, the **nine-point circle** was actually discovered by Karl Wilhelm Feuerback (1800–1834). He published a pamphlet in which he displayed the nine-point circle and also proved that it was tangent to the inscribed and the three escribed circles of the given triangle. This fact is know as Feuerbach's Theorem and in German the nine-point circle is know as **Feuerbach's Circle**.

To construct the nine-point circle, first draw a good-sized triangle ABC in the middle of a large sheet of paper. It is perhaps easier to use an acute triangle, but a little more can be seen with an obtuse one. In your drawing and constructions, you must be as accurate as possible in order to see how everything fits together: the circumcenter, the inscribed and escribed circles, the altitudes and the midpoints of the sides and the centroid. Use colors to distinguish the various figures, e.g., green for the triangle.

1. Find the **circumcenter** O of the triangle.

2. Find the **orthocenter** H by dropping the **altitudes** from each vertex onto the opposite sides. Let the altitude from A onto BC be labeled H_1. The altitude from B onto AC be H_2 and the altitude from C onto BA be H_3.

3. Find the **midpoints of the distances from the orthocenter** H to the three vertices A, B and C. That is, find the midpoints of HA, HB and HC. Label these midpoints C_1, C_2 and C_3.

4. Find the **midpoints of the sides** and label them M_1, M_2 and M_3, the midpoints of sides BC, AC and BA respectively. (These midpoints were produced in Step 1, when the perpendicular bisectors were drawn.) Connect the midpoints with the opposite vertices to find the **centroid**, the "center of gravity," and label this point G.

5. Find the **incenter** and the three **excenters** and draw the **inscribed circle** or **incircle** and the three **external circles**, the **ecircles**.

6. Draw the line OH and find its midpoint, F. This point is the center of the nine-point circle. Note that the centroid G also lies in this line! In fact HG = 2 times GO. This line is know as the **Euler Line** of the triangle.

7. Draw a circle with center F which passes through the nine points: M_1, M_2 and M_3; C_1, C_2 and C_3; and H_1, H_2 and H_3. This is the **Nine-point Circle** or **Feuerbach Circle**. This circle is also tangent to the inscribed and three excribed circles, and so could also be called the **Thirteen-point Circle**.

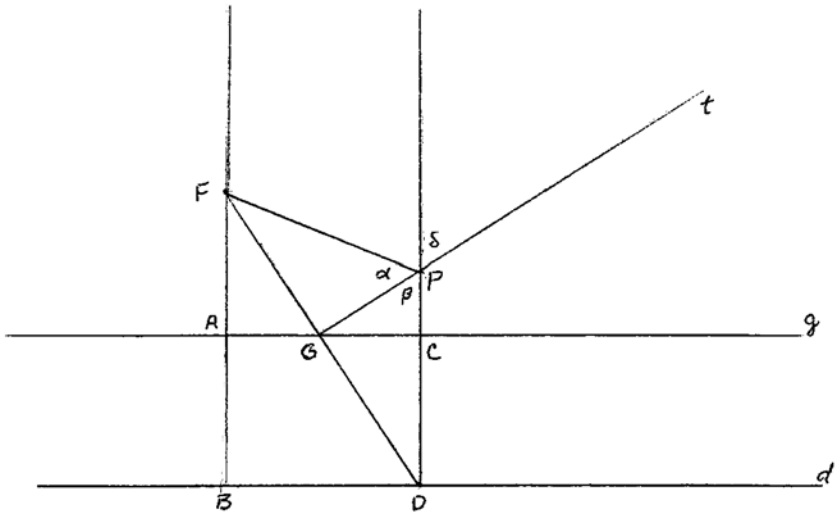
Additional Construction:

8. Let H_2H_3, H_3H_1 and H_1H_2 intersect the opposide sides BC, AC and AB respectively in points P_1, P_2 and P_3. Then P_1, P_2 and P_3 lie on a line know as the **polar axis** of the triangle ABC and the polar axis is perpendicular to the Euler Line. If the nine-point circle and the circumcircle intersect, then the polar axis is the line of the common cord of these two circles. Draw the circumcircle to show this. The circle with HG as diameter is called the **orthocentroidal circle** of the given triangle.

9-5

Proof of Tangent Line Construction for a Parabola

Rays from the Focus reflected from the generating line are tangent to the Parabola.



F is the Focal point or Focus.
A is the Apex.
d is the directrix.
g is the generating line, which is // to d.

Let D be any point on d.
Draw FD intersecting g in G.
Draw FA,the axis, intersecting d in B.
Construct line t $\perp$ FD at G.
Construct the perpendicular to d at D, intersecting g at C and t at P.

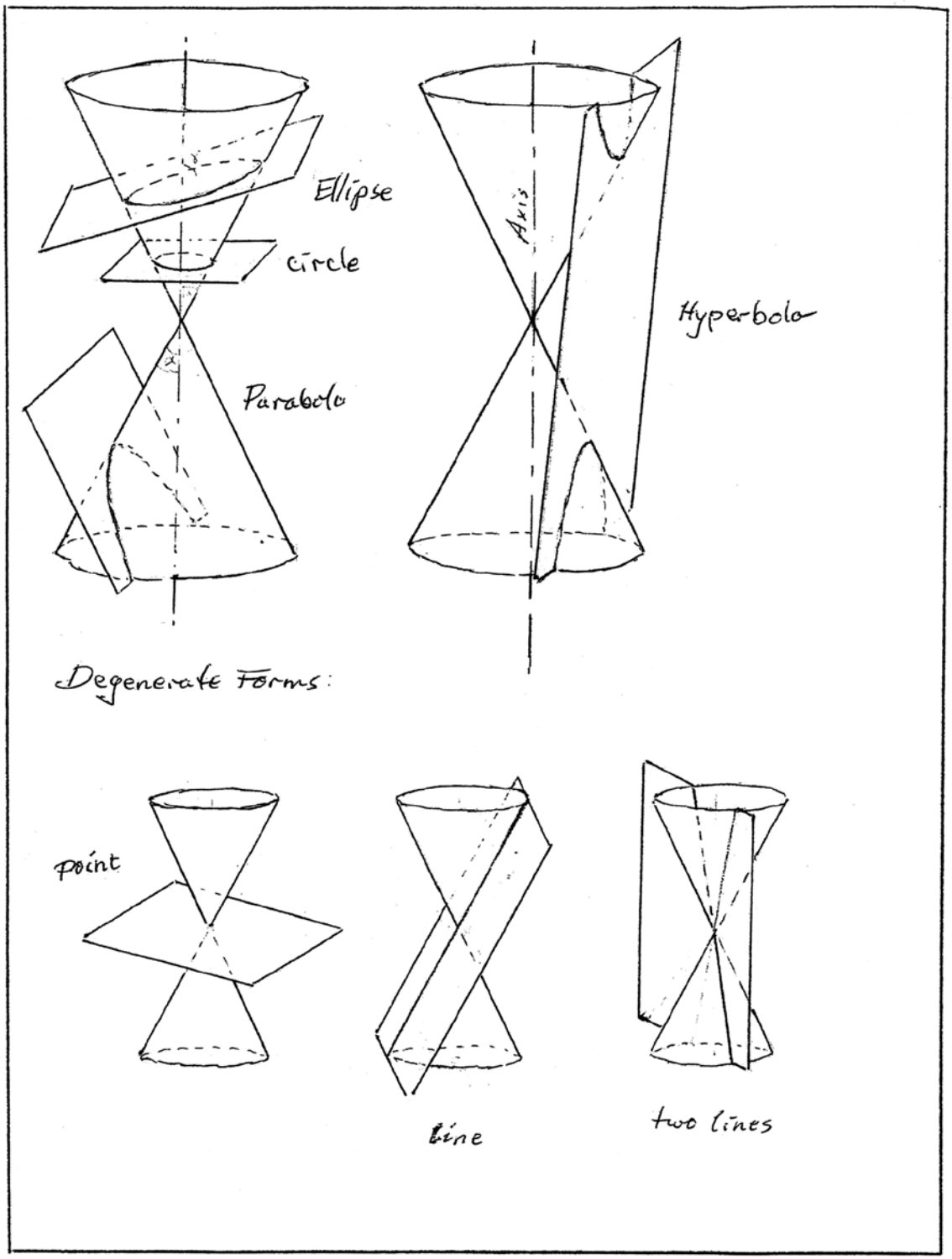
FA = AB by definition
AB = CD, since g//d
Thus FA = CD
Therefore Δ FAG = Δ CDG
So that FG = GD

<FGP = < DGP = right angle, since FD $\perp$ GP by construction
GP = GP

Therefore ΔFGP = Δ DGP

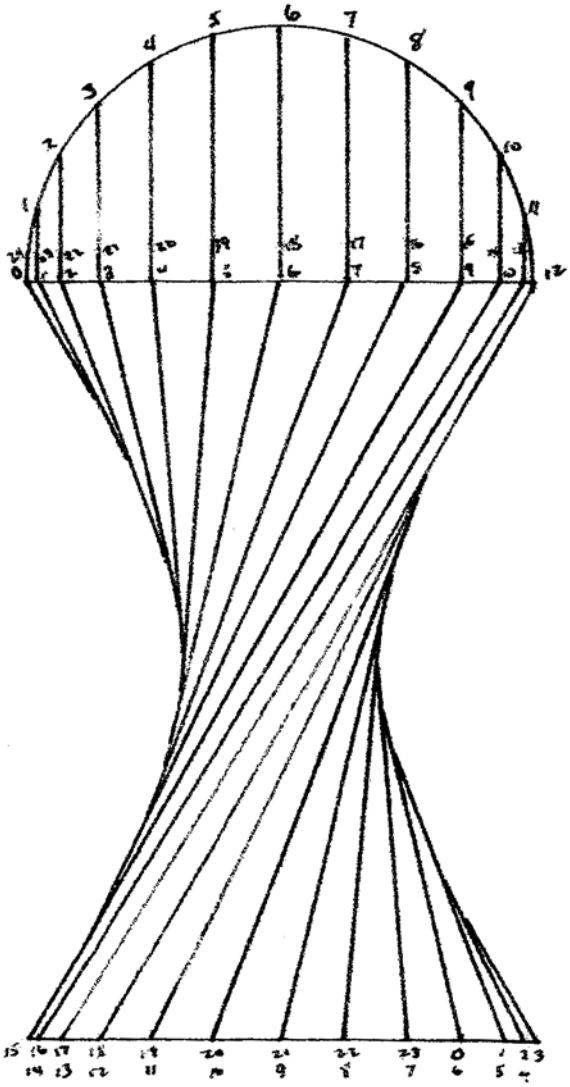
Therefore $< \alpha = < \beta$
$< \delta = < \beta$ (vertical angles)
Therefore $< \alpha = < \delta$
This means that rays from F to P are reflected parallel to the major axis, so P must lie on the parabola.

Conic Sections: Sections of a Cone



9-7

Hyperboloid of One Sheet



Drawing Cassini Curves

The locus of all points, the product of whose distances from two fixed points is constant is a Cassini Curve.

Let r_1 and r_2 be the distances from the two points.

Casssini Curves can be satisfactorily drawn using a separation of the fixed points equal to 10 cm and radii of (in cm):

Radii for the Cassini Curves (cm) using $r = 5x(1.2)^m$ letting m =... -2, -1, 0, 1, 2, ...

1.4
1.67
2.01
2.41
2.89
3.47
4.17
5.0
6.0
7.2
8.64
10.67
12.44
14.93
17.92

Appropriate curves of the different types appear using the following values of k for the product $k = r_1 r_2$:

Two Eyes	k = 17.35
Lemniscate	k = 25
Peanut	k = 30
Racetrack	k = 43.2
Oval (Roundtable)	k = 90

9-9

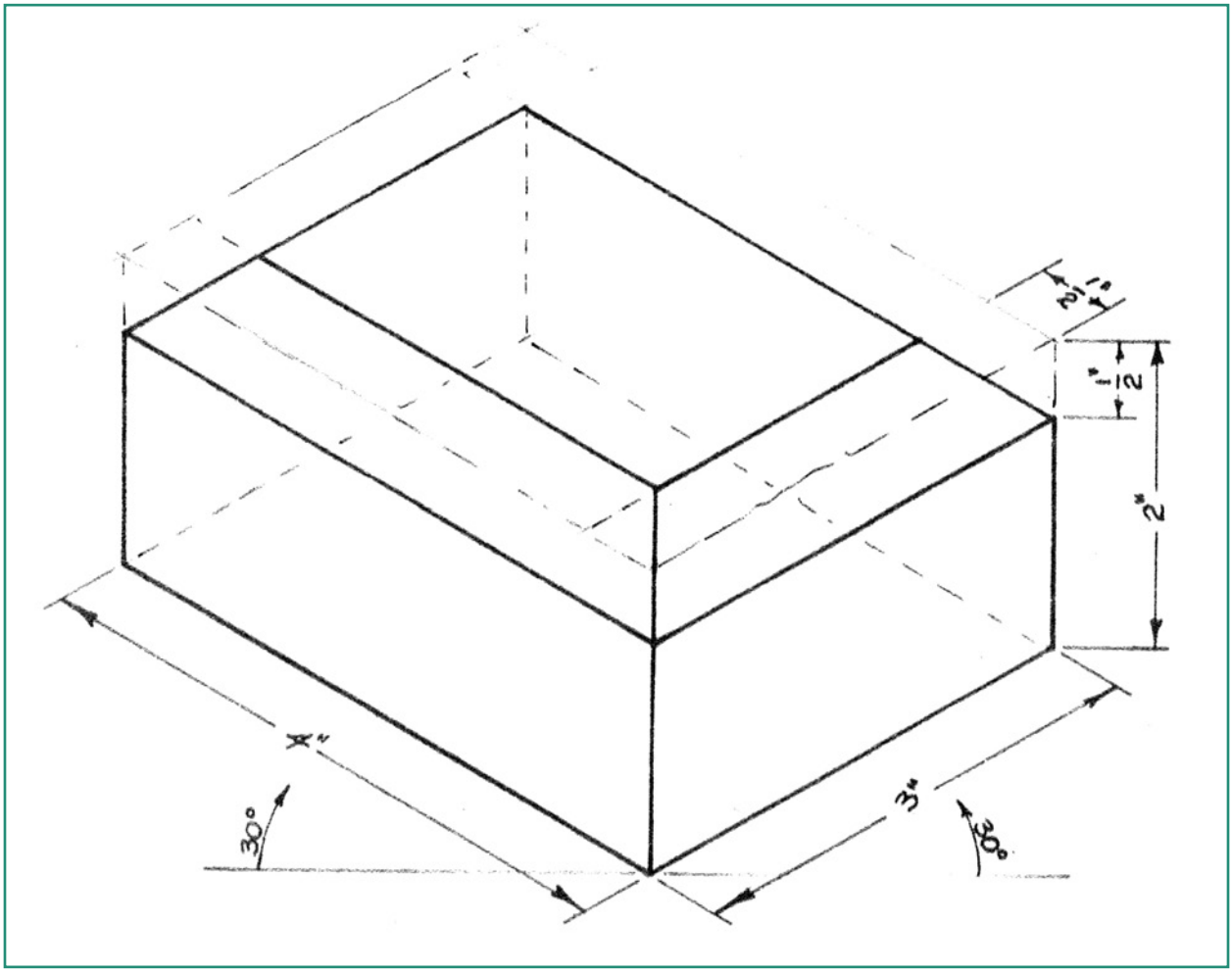
Conchoid of Nicomedes

1) Set out a page with the long side horizontal. Draw a base line about 4 cm from the bottom border and a fixed point 6 cm above the center of the line. Mark off every 2 cm on the line and draw (light) construction lines from these points through the fixed point.

2) Now mark off every 2 cm on the construction lines, beginning at the base line and passing through the fixed point. Connect the points which lie equidistant from the baseline in smooth curves. Color these curves in an appropriate color sequence to show the developing figure. These curves are "Conchoids of Nicomedes."

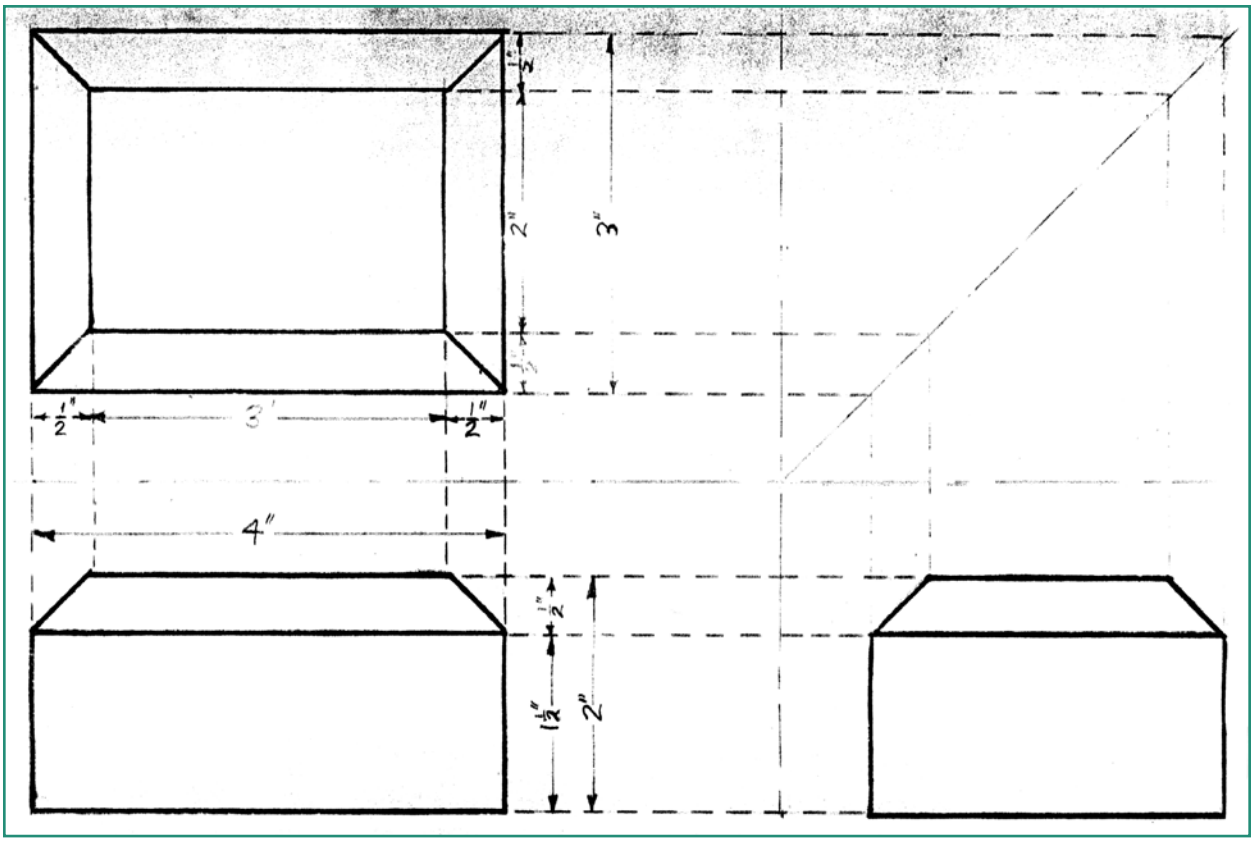
3) Draw an equilateral triangle in the center of another sheet of paper. The triangle should be as large as fits conveniently, but leave some space around it. To facilitate the drawing, let the sides be an exact number of centimeters, for example, 14 cm. Mark off every centimeter on the sides. Find the center of the triangle. Construct conchoids using the center as the fixed point and each side in turn as the baseline. The figure will be most impressive if you stop at the angle bisectors. Color the conchoids to show the development of the figure.

Isometric of Beveled Block

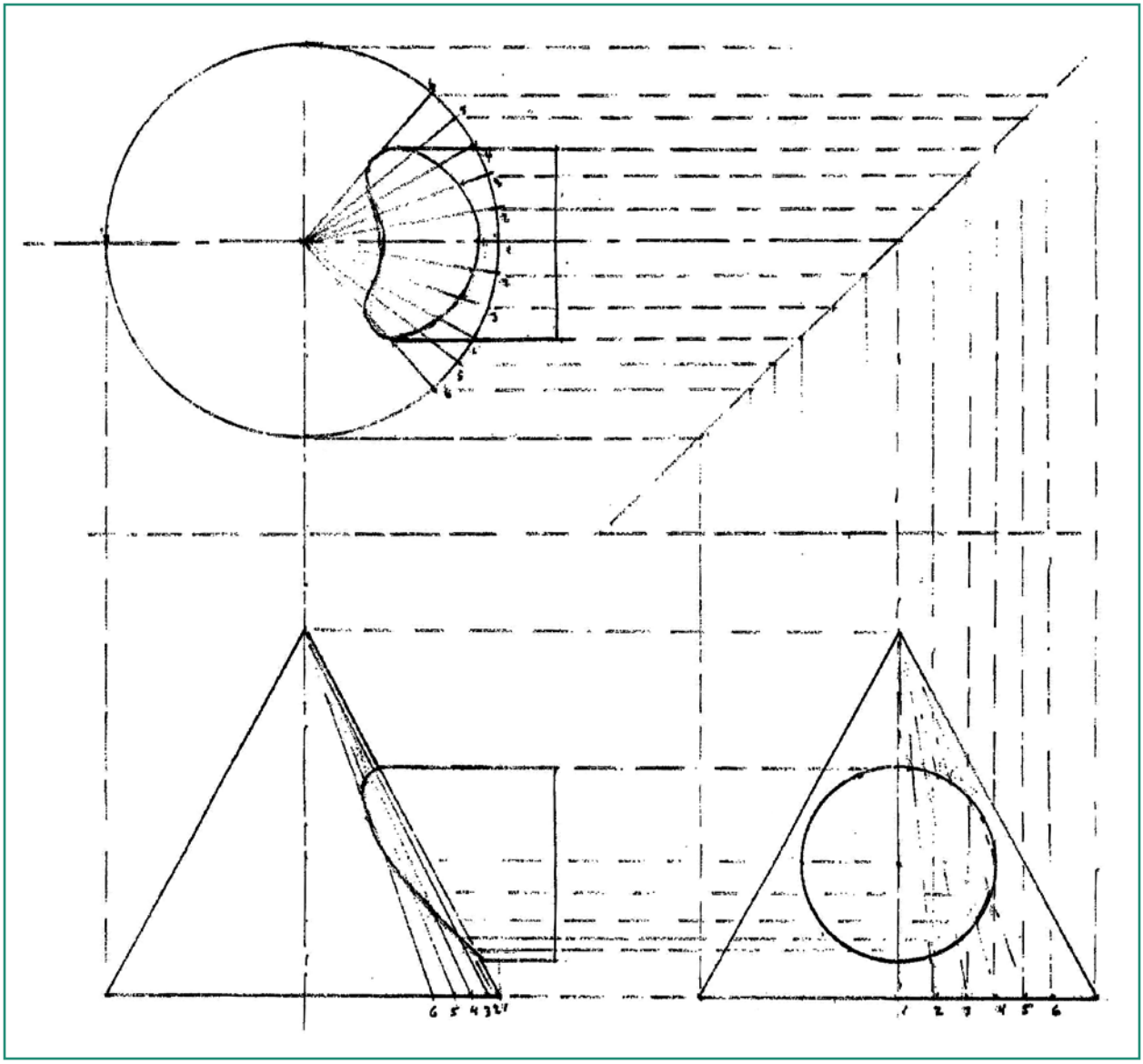


9-11

Orthographic Projection of Beveled Block

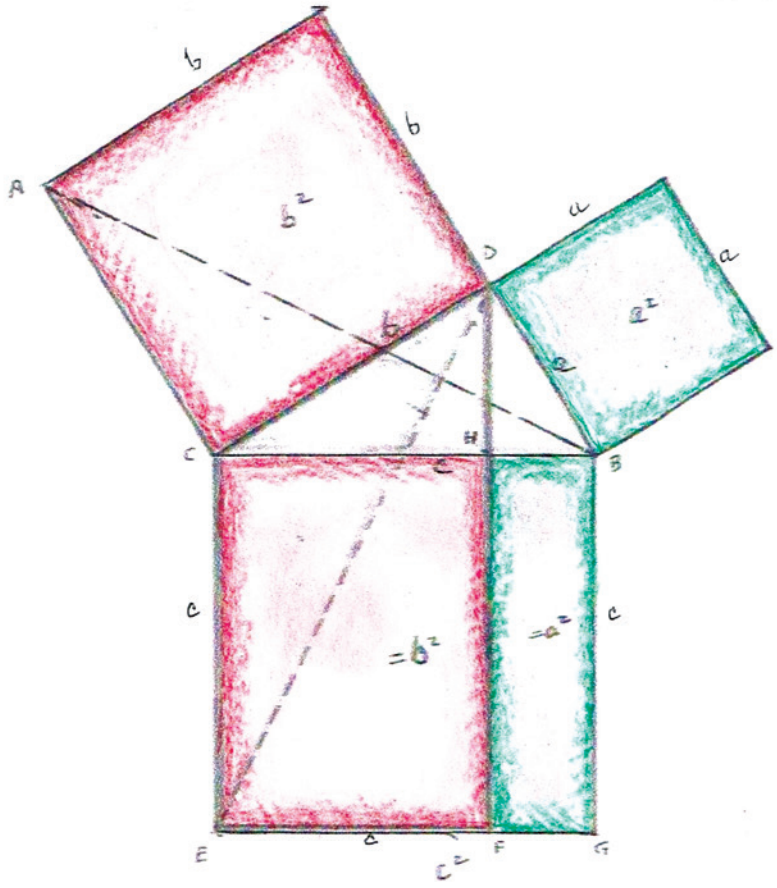


Intersection of a Cone and a Cylinder



9-13

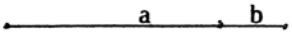
The Pythagorean Theorem: A Proof



AC = CD
CB = CE
<ACB = <ECD Right angles + common angle, <BCD
ΔABC $\cong$ Δ DEC
Thus Area ΔABC = Area Δ DEC
Area ΔABC =½ $\overline{AC}$•b = ½b^2
Area ΔDEC =½ $\overline{EF}$ • CE
Thus b^2 = EF • CE
A similar argument shows that a^2 = FG•GB
Thus c^2 = a^2 + b^2

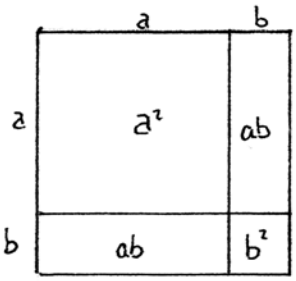
Powers of Binomials

$(a + b)^1 = a + b$



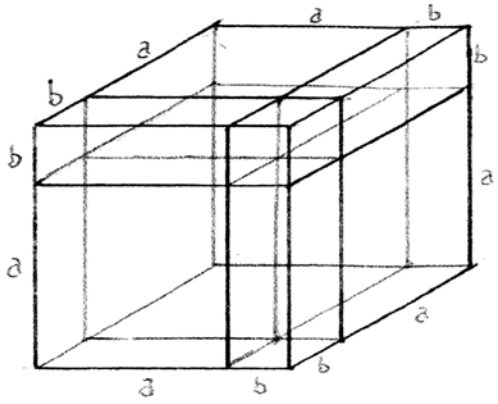
$(a + b)^2$

$= a^2 + 2ab + b^2$



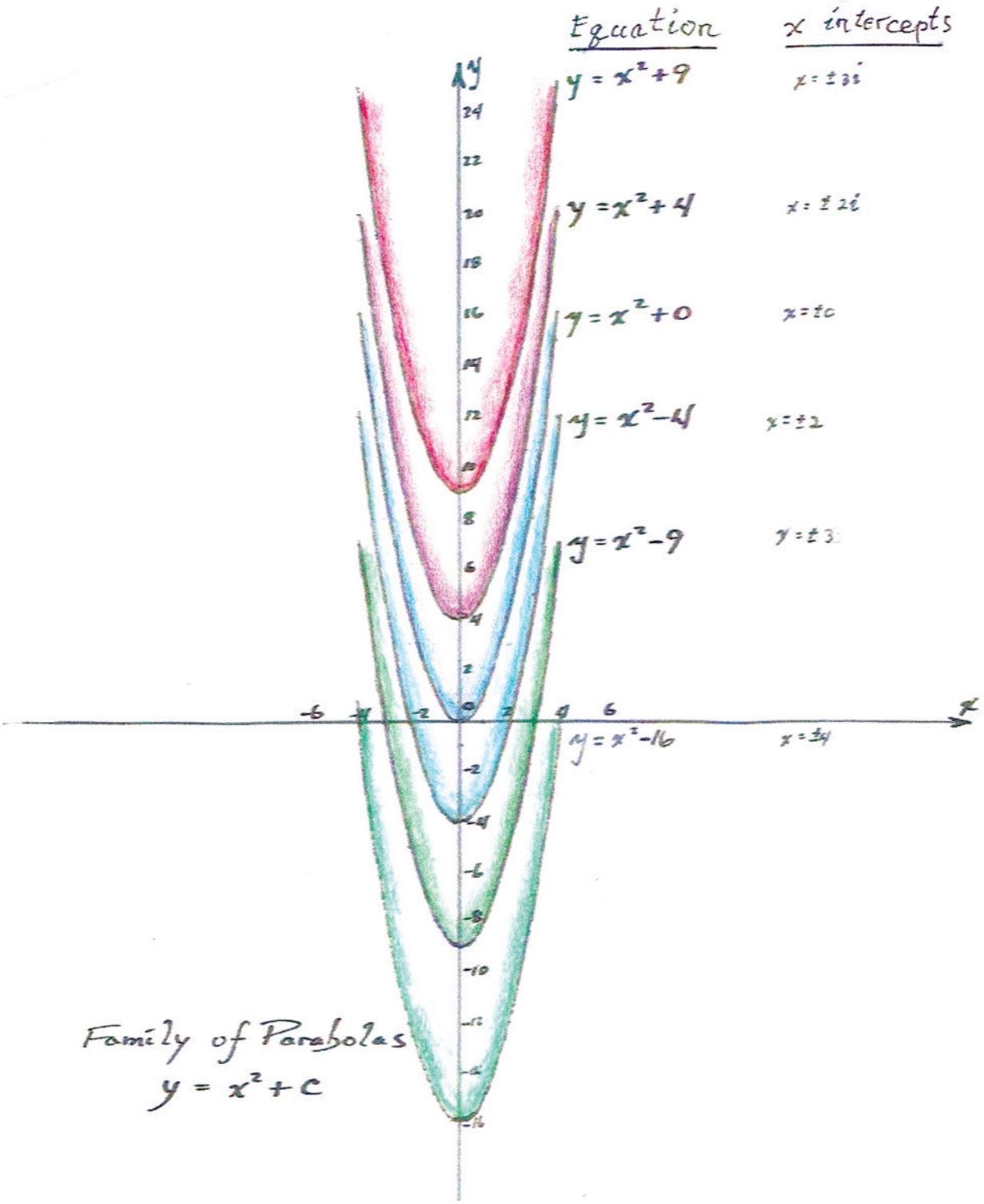
$(a + b)^3$

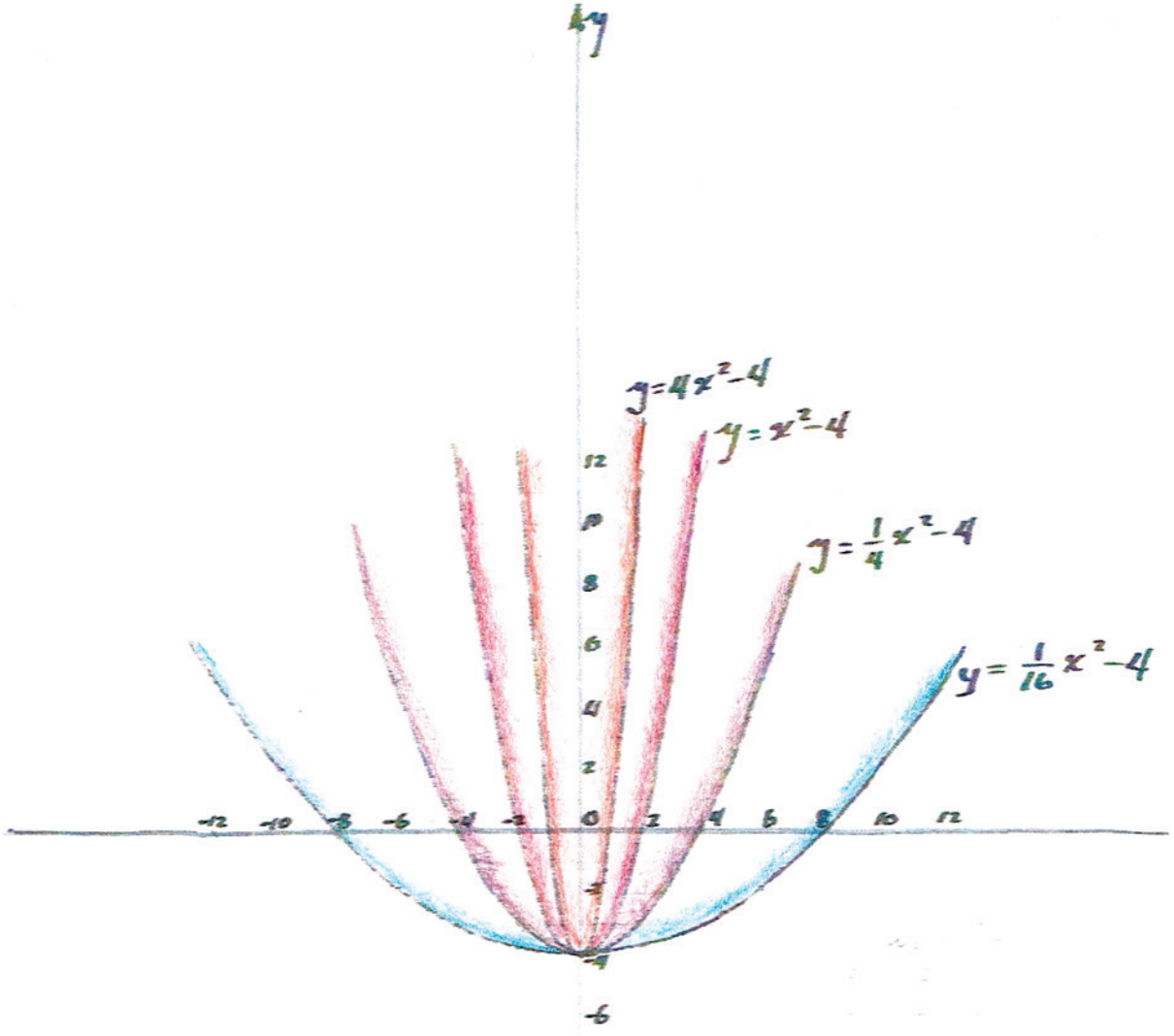
$= a^3 + 3a^2b + 3ab^2 + b^3$



9-15

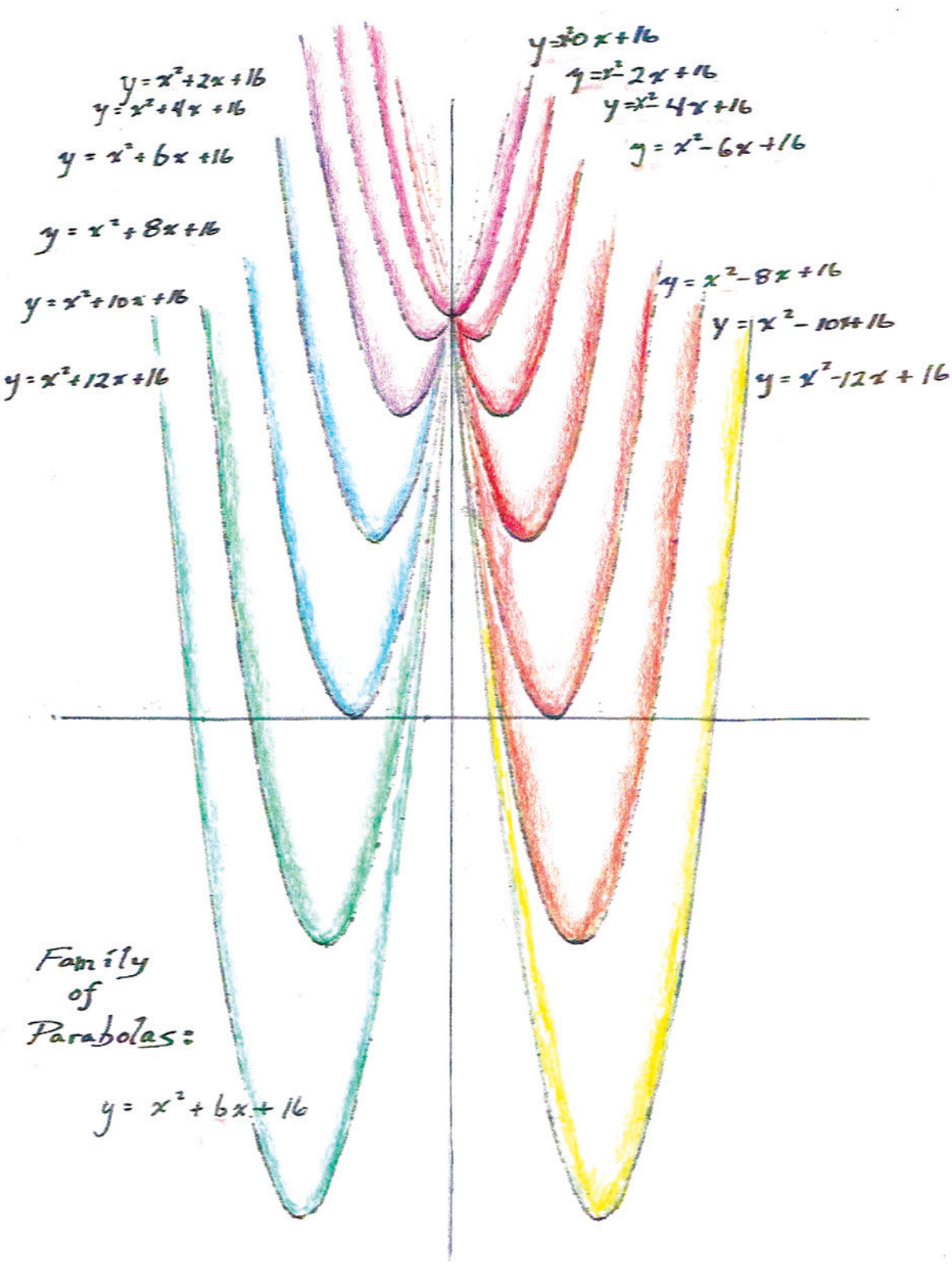
Families of Parabolas





Family of Parabolas

$y = ax^2 - 4$


y = x² + 2x + 16
y = x² + 4x + 16
y = x² + 6x + 16
y = x² + 8x + 16
y = x² + 10x + 16
y = x² + 12x + 16
y = x² - 2x + 16
y = x² - 4x + 16
y = x² - 6x + 16
y = x² - 8x + 16
y = x² - 10x + 16
y = x² - 12x + 16
Family of Parabolas:
y = x² + bx + 16

Quadratic Equation Solved Three Ways

I $x^2 - 4 = 0$

Solution by Graphing:

$x = +2, -2$

Solution by Factoring:

$(x + 2)(x - 2) = 0$

$x + 2 = 0$ or $x - 2 = 0$

$x = -2$ $\quad$ $x = 2$

Solution by Completing the Square:

$x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

Solution by Formula:

$$x = \frac{\pm\sqrt{16}}{2}$$

$$= \frac{\pm 4}{2}$$

$$= \pm 2$$

II $\frac{x^2}{4} - 4 = 0$

Solution by Graphing:

$x = \pm 4$

Solution by Factoring:

$$\left(\frac{x}{2} - 2\right)\left(\frac{x}{2} + 2\right) = 0$$

$\frac{1}{2}x - 2 = 0$ or $\frac{1}{2}x + 2 = 0$

$\frac{1}{2}x = 2$ or $\frac{1}{2}x = -2$

$x = 4$ or $x = -4$

Solution by Completing the Square:

$$\frac{x^2}{4} = 4$$

$x^2 = 16$

$x = \pm\sqrt{16}$

$= \pm 4$

Solution by Formula:

$$x = \frac{\pm\sqrt{4}}{\frac{1}{2}}$$

$$= \frac{\pm 2}{\frac{1}{2}}$$

$= \pm 4$

III $x^2 - 6x + 16 = 0$

Solution by Graphing:

There is no solution.

Solution by Factoring:

The expression does not factor.

Solution by Completing the Square:

$x^2 - 16x + 9 = -7$

$(x - 3)^2 = -7$

$x - 3 = \pm\sqrt{-7}$

$x = 3 \pm \sqrt{-7}$

$= 3 \pm \sqrt{7}\,i$

Solution by Formula:

$$x = \frac{6 \pm \sqrt{36 - 64}}{2}$$

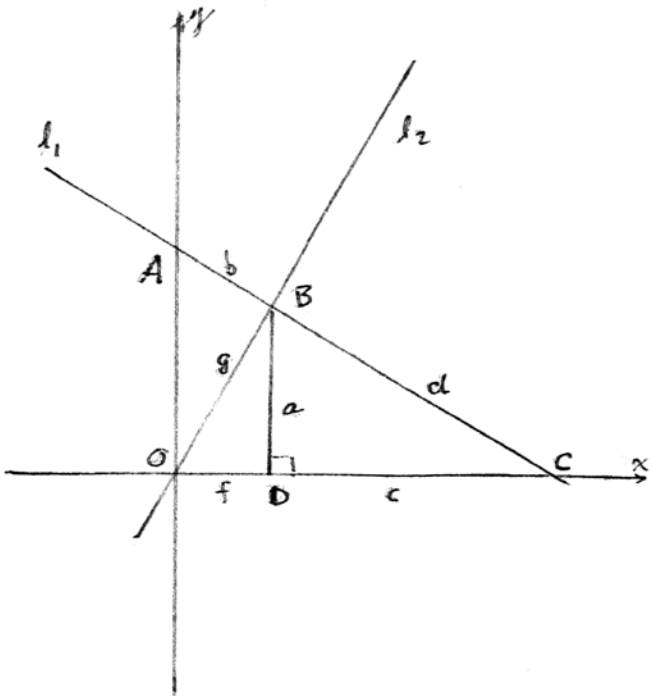
$= 3 \pm \sqrt{7}\,i$

9-17

Theorem: Two Perpendicular Lines and Their Slopes

To prove: If two lines have slopes whose product is -1, then they are $\perp$ and vice versa.

I. Given l_1 and l_2 with slopes m_1 and m_2 intersecting in B and $l_1 \perp l_2$. For convenience let l_2 pass though the origin. Drop a perpendicular to the x axis from B intersecting the axis at D. Line l_1 intersects the y axis and the x axis in points A and C respectively. Line l_2 intersects the axes at the origin.



1. Δ CBO and Δ OBA are right triangles.
2. Δ ABO $\sim \Delta$ OBC $\sim \Delta$ AOC
3. $m_1 = -e/c$
4. $m_2 = a/f$
5. $= c/e$
6. $1/m_2 = e/c$
7. $m_1 = -1/m_2$
8. $m_1 m_2 = -1$

II. Given $m_1 m_2 = -1$

1. $m_1 m_2 = -1$
2. $m_1 = -1/m_2$
3. $m_1 = -e/c$
4. $m_2 = a/f$
5. Δ BOD $\sim \Delta$ CDB $\sim \Delta$ BAO
6. $a/f = d/g = c/a$
7. $<$ ABO $=$ $<$ OBC
8. Therefore $<$ ABO and $<$ OBC are right angles.
9. Thus $l_1 \perp l_2$ Q E D

Grade 10 Math

After the excitement, chaos, and struggle of the 9th Grade, students are ready for some balance, peace, clarity. History returns to the ancients. Color returns to Art. In Chemistry the focus is on aqueous systems, where water seeks a plane or perfect sphere and mediates between extremes of acid-base or crystals and solution. In Physics, students should develop a confidence in their thinking. This thinking develops logically, a process begun in classical Greece.

In mathematics the Algebra block develops the quality of balance in beauty, the Golden Section, Golden Rectangle. This ratio turns up in the limit for the Fibonacci sequence. Other sequences have their own rules and relations. The Geometry block takes geometry into the practical sphere of Surveying, applying trigonometry, which would be the practical block of this year. The track class is appropriately Plane Geometry, developed in ancient Greece and systematized by Euclid. Building everything up from axioms and postulates produces a formal, logical structure which is also practical. In computer science students learn about number bases, logic and machine and higher level programming.

10th Algebra Block

Pythagoras was the father of mathematics. The four branches of mathematics (astronomy, geometry, arithmetic and music), arise from the four possibilities of movement and magnitude (movement and rest, discrete and continuous magnitudes). (See Appendix) Each has its corresponding progression, namely, arithmetic, geometrical, astronomical and musical. For Pythagoras the universe was a great monochord, one end attached to absolute spirit, the other to matter: from heaven to earth. It was stimulated by the motion of the planets, each sounding its own note.

There is a wide variety of counting methods from around the world. Also some animals have been discovered to have primitive counting systems. Many numbers have special meanings. For Pythagoras the first four were especially significant, and all the rest were derived from them.

The harmony of the Golden Section deserves a special place in this block. Its appearance in art, architecture, shapes of many familiar objects from paper and books to windows and houses is quite remarkable. Geometric construction and calculation could follow. Then the surprising limit of the Fibonacci series and the appearance of this series in plant leaves and sunflowers (See, e.g., Jacobs, also Ian Stewart, "Daisy, Daisy, Give Me Your Answer, Do," *Scientific American* January 1995, pp. 76–79), which may lead into

the exploration of series in general: in particular, arithmetic and geometric, also spirals. (Examples in Appendix) Spirals should be drawn: Fibonacci, arithmetic, geometric/ logarithmic.

Possible examples of geometric progressions from music and literature:

Pete Seeger, "We'll All Be A-Doubling"
Ellis Parker Butler, "Pigs Is Pigs"

Practical uses of logarithms: decibels, pH values. See Appendix.

In connection with the SAT, students should have the opportunity to practice the relations of terms and sums. Number bases would be a good next step: ancient and decimal, then the ones associated with computers: binary, octal, hexadecimal; and conversion. (Cf *Mathematical Ideas*.) The next step toward programming is some basic logic: symbols, statements, truth tables, circuits, and then an introduction to machine and higher level programming.

Surveying

If the school is new to the surveying block, someone experienced needs to be found, a visiting teacher, local retired surveyor... Equipment needs to be assembled, bought or borrowed. Traditional transits should be used instead of modern ones.

Mathematics Track Class: Plane Geometry

1. Review of basic geometric relations: points, lines, planes, their intersections. Geometric figures, perimeter, areas.

2. Proofs in Geometry: Axioms, Postulates, Theorems, Proofs: the logical foundation of Euclidean geometry and other mathematical systems. Construction of a formal proof.

The necessity of proofs. Before students begin to work with formal proofs, it is well for them to experience an example of an apparent rule which breaks down with a little more probing. A simple example is the relation between the number of areas of a circle produced by chords according to certain rules. Some rules produce a simple general rule, some do not.

A. Circles divided into the maximum number of areas by chords. Students should draw a series of circles. For the first circle, let there be one points and no chords resulting in one area. For the second circle, let there be one chord intersecting the circle in two points, resulting in two areas. For the third circle, there are 3 chords, 6 points and 4 areas, and so

on. An apparent rule appears, namely, the number of areas equals the last number of areas plus the new number of points. This rule is valid.

B. Now let them try another series: Circles are divided by chords joining points on the circumference. For one point, there is one area, for two points, two areas, for three points, four areas, for four points, 8 areas, for 5 points, 16 areas. Let n = the number of points, k the number of areas. Then for n>0, a rule appears, namely: k= 2n-1. However upon trying the next number of points, namely, 6, the sum turns out to not be the expected 32, but only 31! For 7 points the discrepancy is even larger. (The numbers are chopped off, so to speak, from the lines of the Pascal triangle: first one number from the fifth row (26 -1), then two numbers from the next (total 3) and so on. Thus mathematicians demand rigorous proofs, not what may seem to be true for several cases.

3. Angles: Complementary, Supplementary, Vertical. Parallel Lines, Transversals and their Angles

4. Triangles, Polygons, Interior and Exterior Angles, Angle Sums
Because of their regular appearance on the SAT, special attention should be given to 30-60-90° triangles and 45° right triangles, also the simple Pythagorean triangles, 3-4-5 and multiples. Problems with multiple triangles enmeshed or attached are often on the SAT.

5. Congruent Figures

6. Similarity, similar triangles, angle bisectors, right triangles

7. Basic trigonometric functions (sine, cosine, tangent) and calculations
While the origin of the word *cosine* is clear enough, that of *sine* and *tangent* are not. A simple diagram shows the origin of the word, Tangent, while the word Sine comes to us on an odd path from the Hindu. These are illustrated in the Appendix, where there is also a poem encapsulating the derivation of the word *sine*.

Using the definitions, students can calculate the trig functions for 30, 60 and 90 degrees using the 45° and 30-60-90° triangles. They can also read them off of 1 cm graph paper lined at 1mm by drawing a 10 cm quarter circle in the lower left corner and adding radii at, say, every 15 degrees with their tangents. From intersections of the radii with the circle, run horizontal and vertical lines to read off cosine and sine respectively. See Appendix.

If students are having trouble remembering what ratios to use for the basic functions, they could try the mnemonic device, "sohkahtoa." Here this stands for: "sine (equals) opposite (over) hypotenuse, c(k)osine equals adjacent (over) hypotenuse, tangent (equals) opposite (over) adjacent."

8. Relations between the functions, identities.

9. Using trig functions.

10. Circles and their tangents.
Concentric, intersections
Inscribed and circumscribed polygons
Central angles and arcs, arcs and chords, inscribed angles
Chords and secants

11. Areas of plain figures, volumes

Note: It may be necessary to include some of the topics suggested for the Algebra block in the Track class.

Computer Science

In Algebra block or Track class: Machine language, higher languages, logic, programming

References

Jacobs, Harold R. *Geometry*
Ray C. Jurgensen, et al. *Geometry*, McDougal Littell
Charles D. Miller, et al. *Mathematical Ideas*

10th Grade Appendices

10-1 Artistic Coloring of Plates
10-2 "The Sine," a poem
10-3 Trigonometric Functions on the Unit Circle
10-4 Measured Values of the Trigonometric Functions
10-5 Some Uses of Logarithms
10-6 The Four Branches of Mathematics
10-7 Golden Rectangles
10-8 Series of Golden Rectangles
10-9 Arithmetic Spirals
10-10 Geometric/Logarithmic Spiral (Nautilus)
10-11 Drawing Geometric Spirals

Artistic Coloring of Plates

Three steps are required to produce an enhanced drawing without making it childish or gaudy:

1. Heavily color just inside the outer line (perimeter) of the figure.

2. With a sharpened, but not sharply-pointed tip, lay the colored pencil tip flat against the paper with the end of the tip on the outer border. With slightly more pressure toward the tip, shade around the inside of the perimeter.

3. Finally use the sharpener to make a few shavings, spread them over the empty center of the figure, and rub gently, harder toward the outside.

10-2

The Sine

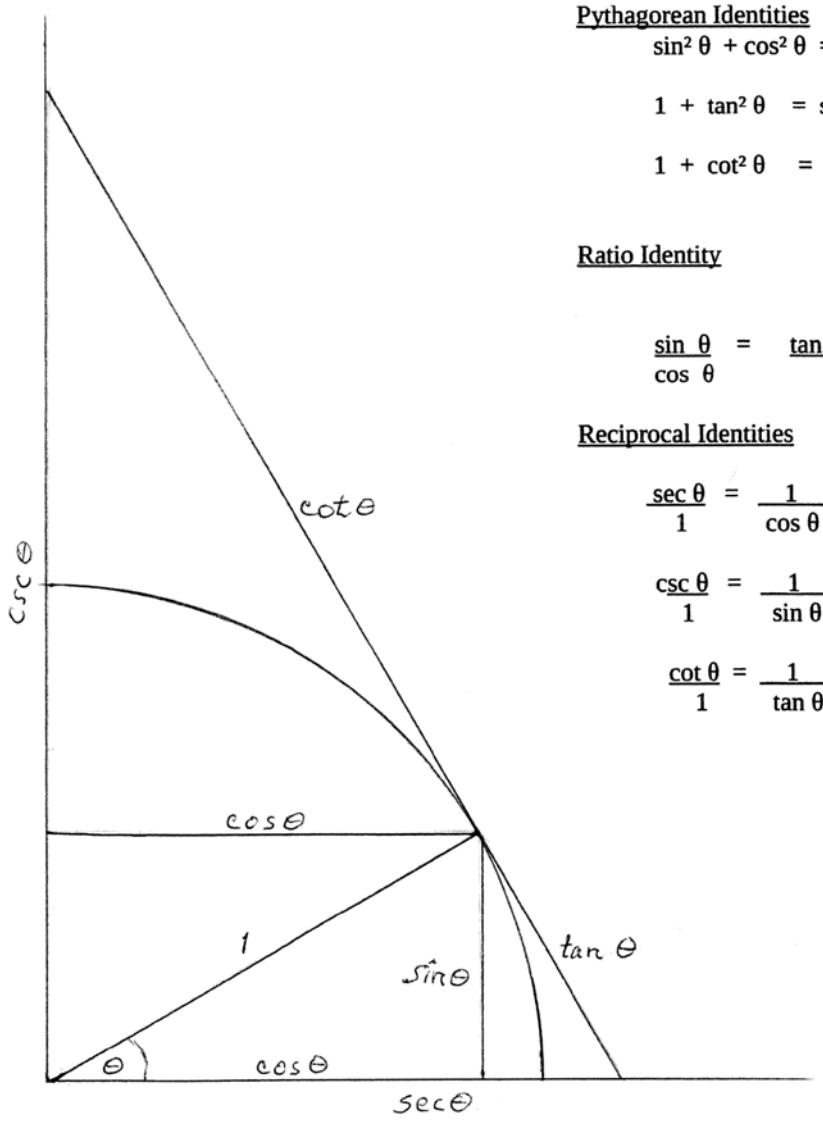
We meet the sine in trigonometry,
Developed out of circular geometry.
The spelling is not what we're used to;
"S I N E" is something to adjust to.
While most words' meanings are quite
reasonable,
The origin of this one is most unseasonable.
The Hindu mathematician, Aryabhata,
Coined the word *ardha-jya*,
Which, although it may not say much to you,
To Hindus meant "a chord cut half way
through."
Later, to *jya* it was abbreviated,
So verbal effort was alleviated.
Now I think this confusing enough,
But then the Arabs added "b," so much fluff,
That a further contraction was in store
To simply "jb," nothing more.

Such nonsense surely could not stay,
So they changed it to the word for "cove"
or "bay."
In Arabic, *jaib* was the word;
In Latin, *sinus* – how absurd!
Thus the sign we carry is not spelled
the same,
Though both are upright in the main,
Which for "sine" is quite commendable,
From such unseemly origins to be
upendable.

We use them both for surveying now,
For roads and lots and fields with cow.
In right triangles sines are found:
If we pick one acute angle, we are bound
To look across to the other side.
Standing upright with considerable pride,
The opposite side may seem of little use,
But when divided by the hypotenuse,
A sine results, whose value lies
'Tween zero and one; and I surmise
That somehow magical mathematical
Or calculation most fanatical
Has transformed and served for luncheon
A triangle formed from a circular
function!

– R.C. Oelhaf

Trigonometric Functions on the Unit Circle



Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Ratio Identity

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

Reciprocal Identities

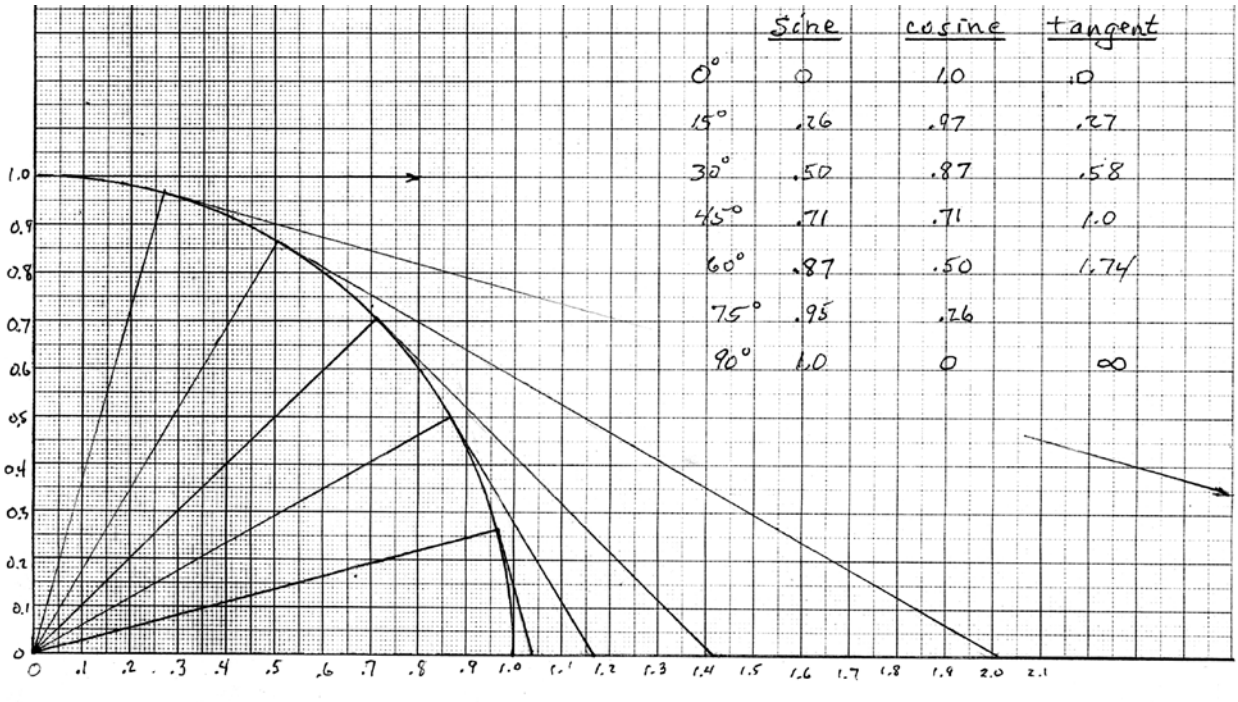
$$\frac{\sec\theta}{1} = \frac{1}{\cos\theta}$$

$$\frac{\csc\theta}{1} = \frac{1}{\sin\theta}$$

$$\frac{\cot\theta}{1} = \frac{1}{\tan\theta}$$

10-4

Measured Values of the Trigonometric Functions



	Sine	cosine	tangent
0°	0	1.0	.0
15°	.26	.97	.27
30°	.50	.87	.58
45°	.71	.71	1.0
60°	.87	.50	1.74
75°	.95	.26	
90°	1.0	0	∞

Some Uses of Logarithms

pH stands for "potency of Hydrogen" and equals the negative logarithm (base 10) of the hydrogen ion concentration in moles per liter.

pH of Some Common Substances

Substance	pH		
0.1*M* HCl	1	Acid	Increasing acidity
Stomach contents	2		
Vinegar	2.9		
Soda pop	3		
Grapes	4		
Beer	4.5		
Pumpkin pulp	5		
Bread	5.5		
Intestinal contents	6.5		
Urine	6.6		
Bile	6.9		
Saliva	7	Neutral	
Blood	7.4	Base	Increasing basicity
Eggs	7.8		
0.1*M* NH_3(aq)	11.1		
0.1*M* NaOH	13		

EXAMPLE: pH of Solution

Find the pH of a solution with H_3O^+ concentration $1.00 \times 10^{-5} M$.

Solving process:

$$\begin{aligned} pH &= -\log [H_3O^+] \\ &= -\log (1.00 \times 10^{-5}) \\ &= -(\log 1.00 + \log 10^{-5}) = -[0 + (-5)] = 5 \end{aligned}$$

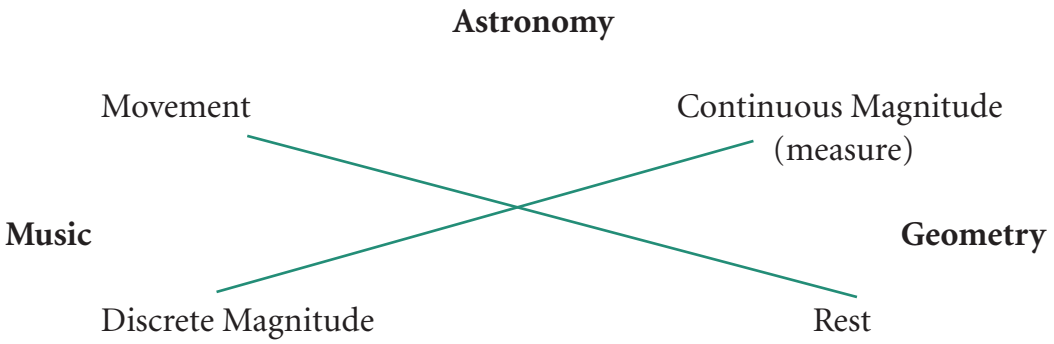
A decibel is one tenth of a bel, a measure of sound intensity with reference to the threshold of hearing.

Source or description of noise	Noise level, dB	Intensity, $W \cdot m^{-2}$
Threshold of pain	120	1
Riveter	95	3.2×10^{-3}
Elevated train	90	10^{-3}
Busy street traffic	70	10^{-5}
Ordinary conversation	65	3.2×10^{-6}
Quiet automobile	50	10^{-7}
Quiet radio in home	40	10^{-8}
Average whisper	20	10^{-10}
Rustle of leaves	10	10^{-11}
Threshold of hearing	0	10^{-12}

10-6

The Four Branches of Mathematics and Their Progressions

Pythagoras: There are four branches of mathematics arising from the two polarities of movement-rest and discrete-continuous magnitude:



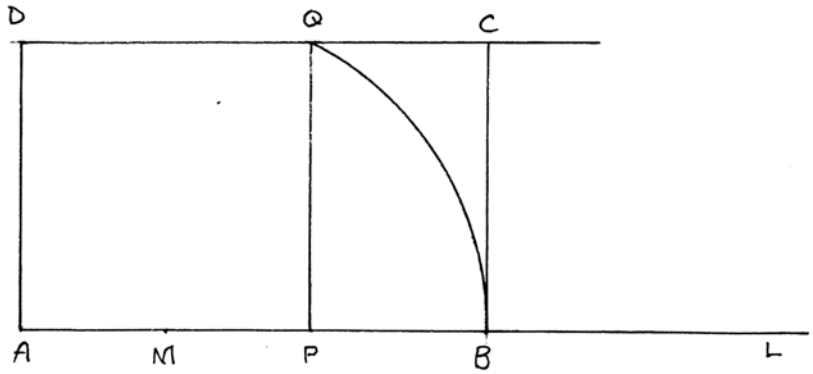
Each of the Four Branches has its own characteristic progressions or series:

1. Arithmetic
2. Geometric
3. Astronomical. The distances are incommensurable; ratios are irrational. The "music of the spheres" is not an earthly, "rational" music
4. Musical
5. (?)Squares, cubes, etc. These are found in the atom; they do not fit an intuitive (human?) picture.

Golden Rectangles

I. Drawing a Golden Rectangle. One way is:

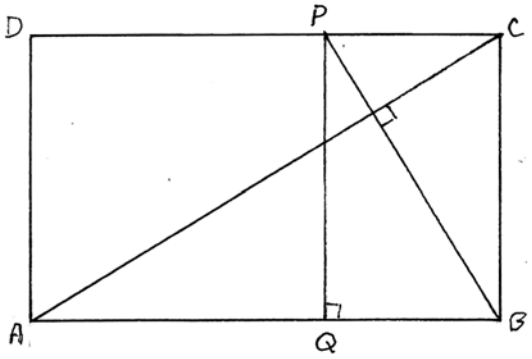
1. Draw line AL.
2. On AL construct square APQD.
3. Find Midpoint M, of AP
4. With center M and radius MQ, draw an arc intersecting AL in B.
5. Construct the perpendicular to AL at B meetng DQ, extended at C.
6. ABCD is a Golden Rectangle.



II. Construct a Gnomen for a Golden Rectangle:

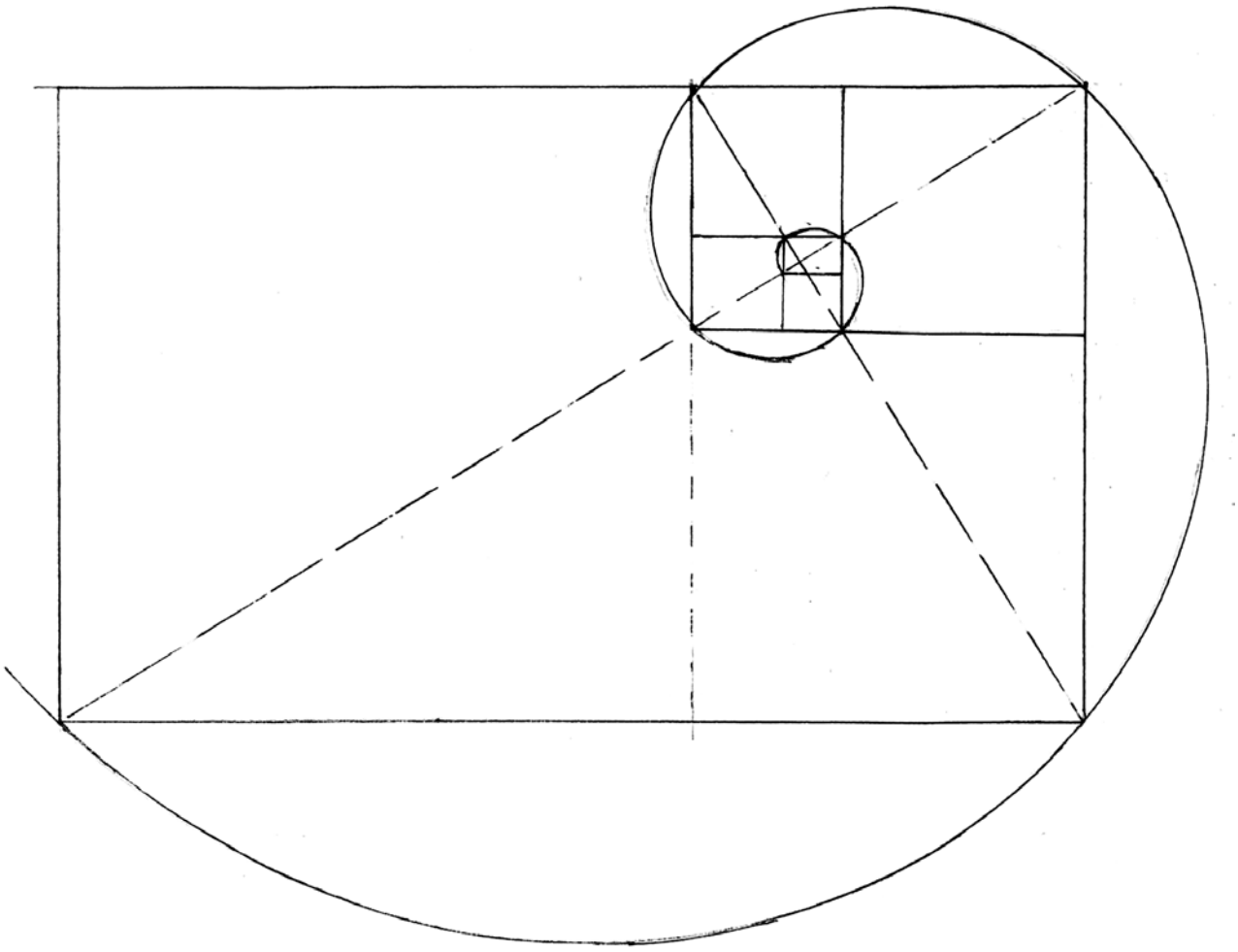
(Note: In mathematics, a gnomen is a shape which, when added to a figure, yields another figure similar to the original.)

1. Given a Golden Rectangle ABCD, draw a diagonal AC.
2. Construct perpendicular to AC from B intersecting CD in P.
3. Draw PQ perpendicular to AB.
4. BCPQ is the required gnomen, a new Golden Rectangle.



10-8

Series of Golden Rectangles: Spiral of Whirling Squares



Arithmetic Spirals



10-10

Geometric/Logarithmic Spiral



Drawing Spirals *(Note: Use polar graph paper.)*

Arithmetic Spiral

Let the radius be 1/20 of the angle in degrees. In other words, go out ½ space for each 10 degrees you go around the center (The formula is $r = \theta/20$, where r is the radius and θ is the size of the angle in degrees.)

Logarithmic/Geometric Spiral

A convenient formula for a logarithmic spiral lets $r = 2^{\theta/120}$ resulting in the following series:

θ(°)	r	θ(°)	r	(θ°)	r
0	1.00	260	4.50	370=10	8.48
10	1.06	2.70	4.75	20	9.00
20	1.12	2.80	5.04	30	9.50
30	1.19	290	5.35	40	10.0
40	1.26	300	5.66	50	10.7
50	1.33	310	6.00	60	11.3
60	1.4l	320	6.35	70	12.0
70	1.50	330	6.72	80	12.7
80	1.59	340	7.13	90	13.5
90	1.68	350	7.55	100	14.3
100	1.78	360	8.00	110	15.1
110	1.89			120	16.0
120	2.00			130	17.0
130	2.12			140	18.0
140	2.24			150	19.1
150	2.37			160	20.2
160	2.52			170	21.3
170	2.66			180	22.6
180	2.83			190	24.0
190	3.00			200	25.4
200	3.17			210	26.9
210	3.36			220	28.5
220	3.56			230	30.1
230	3.78			240	32.0
240	4.00			250	34.0
250	4.25			260	36.0 ...

Grade 11 Math

Students at this age are ready to seek the truth, to get to the heart of the matter. In particular they want to be clear personally about the questions: "Where do I stand in relation to the cosmos and the sub-microscopic world?" and "How I am to set my course in life?" Thus blocks include Music probing the internal nature; Astronomy, probing the depths of the cosmos; Chemistry, probing into the inner depths of material; Physics of electricity and magnetism, a hidden world; Genetics, Parzival.

In Mathematics the geometry block is Projective Geometry which takes us to the infinite periphery and the infinite center. A second mathematics block on Spherical Trigonometry and Navigation would be ideal. Because of pressure from other subjects, it is not always possible to schedule a second mathematics block, so these matters would need to be slipped into the track class.

Track Class

Mathematics now pushes all the elements of earlier courses deeper and further: exponents and logarithms, exponentials, analytical geometry including conic sections, imaginary and complex numbers. The course culminates with the surprising Euler Formula, which unites many elements of the year's work.

Projective Geometry

Projective Geometry takes us to the outer reaches of space, in fact to the infinite, through it and back. It was always surprising to me how easily students took to these ideas, much more easily, it seemed to me, than their elders. And with the concept of pole and polar, we pass from the infinite reaches of space to the infinite depths of the center.

A few references of many excellent ones:
Lawrence Edwards, *Projective Geometry*
____, *The Vortex of Life*
Arnold Bernhardt, *Projective Geometrie*
Amos Francescelli, *Mathematics in the Classroom: Mineshaft and Skylight*
Harry Kretz, *Projective Geometry*

Topics would include:
History (remarkable development of projective geometry by French prisoner)
Projections

Invariance in Euclidean and Projective Geometry. Incidence
Point, Line and Plane at Infinity
Elements and their relations, Duality in plane and space
Theorems of Pappus, Desargues, Pascal and Brianchon
Perspectivities, Projectivities
Projective linewise and pointwise ellipses
Collineation, Step measure, path curves, living forms
Involution, Pole and Polar

Note on Path Curves

To obtain path curves in two dimensions, draw a triangle and run hypergeometric step measure along two sides. Connect the points with the opposite corner. This forms a net. Connecting diagonal points and continuing diagonally across the triangle produces path curves. The curves can be extended beyond the triangle to bring the whole plane into motion. It there is no room for these drawings in the projective geometry block, they can be done in a 12th Grade Mathematics block or as a special topic in the 12th track class, as an illustration of mathematics at the foundation of the real world.

A simple life form in the shape of a pine cone can be obtained from laying out two different geometric series on two parallel lines from a common central axis. That is, we are using a triangle with two infinite sides. If the series are different, an egg or pine cone or bud shape appears when the diagonal sequences are drawn and the form reflected about the finite side. Spirals can be added. See Appendix for convenient geometric series and an example.

Spherical Trigonometry

Since 11th Graders are wondering how to set their course in life, it is well to include at least some basic spherical trigonometry. For convenience, the coordinates of some major cities are listed in the Appendix.
Reference: John A. Graham and Robert H. Sorgenfrey, *Trigonometry with Applications*

Mathematics Track Class: Algebra II

This is the preparation for advanced mathematics and calculus. Major topics include:

- An intensive review and extension of Algebraic forms and operations
- Analytic geometry (from the viewpoints of both Descartes and Fermat, asking respectively, What is the formula for this line? and What is the form for this equation?)

Exponentials and logarithms, e.g., Students should practice calculating with logarithms to better understand them. Although these calculations went out with the computer, still we learn how to do them, just as long division is learned in the lower school. Exponential functions quickly take us to unimaginably large and small quantities.

Trigonometric identities in all quadrants, radians, graphs, sum and difference, double and half angles

Note: Included in the Appendix are several simple geometrical proofs/demonstrations of trig identities. While textbook writers often prefer algebraic derivations, I find geometrical easier to follow and thus more convincing.

In connection with analytic geometry, relate the story of Descartes' discovery while lying in bed watching a fly walk across the ceiling. The monks in the monastery where Descartes was studying as a youth allowed him to sleep in in the morning, which we now know is in accordance with the circadian rhythm of young people. What great discoveries are we missing because we force teens to drag themselves out of bed for early morning classes?

With imaginary and complex numbers, we penetrate beyond the physical world, or penetrate more deeply into it. It is generally reassuring to realize that mathematicians themselves initially rejected each extension of mathematics to new kinds of numbers, in this case, the so-called imaginary numbers. Take a bicycle tire: Viewed from the front of the bike, we may see the valve or pedal moving up and down, slowly at the top, faster as it descends, then again more slowly. We can also imagine the tire viewed from the side of the bike. Of course in this case we can also turn the tire and look at the side view.

A convenient way to recall the trig functions which are positive in the four quadrants is the mnemonic device, "All students take calculus." The first letters of these words are a, s, t and c. Placed in the four quadrants in sequence from I to IV, they indicate that in the first quadrant "all" functions are positive, in quadrant II, only the *sine*, thus "students," in the third only the tangent, thus "take" and in the fourth quadrant, only the cosine, with first letter to match calculus.

Euler's Formula

The various topics of Algebra II may appear simply a list of unrelated material. However a formula attributed to Euler ties them together beautifully. Steiner recommended bringing the 11th Grade mathematics program to a climax with Euler's Formula. This Formula can be simply stated, noting that it would be proved the following year in calculus, where there are easy proofs, such as using the Taylor series expansions of e, sine and cosine. To baldly

state the relation, though possible, does seem somewhat unsatisfactory. Richard Feynman, wishing to use the relation without a formal calculus proof, describes a way using only the functions already studied, namely, exponentials, sine and cosine and complex numbers. Miraculously the formula appears without calculus. See Feynman, *Lectures on Physics*, pp. 22-4 to 22-10. The resulting formula, which Feynman calls "the most remarkable formula in mathematics" and "our jewel," is:

$$ei\,\theta = \cos\theta + i \sin\theta$$

This formula makes many physics and engineering calculations far easier, for the exponential form can often be substituted for the trigonometric functions. And it also links the major areas of 11th grade mathematics, namely, exponentials and logarithms and trigonometry and complex numbers. When $\theta = \pi$ the formula reduces to:

$$ei\pi = -1 + 0 \text{ or}$$

$$ei\pi + 1 = 0$$

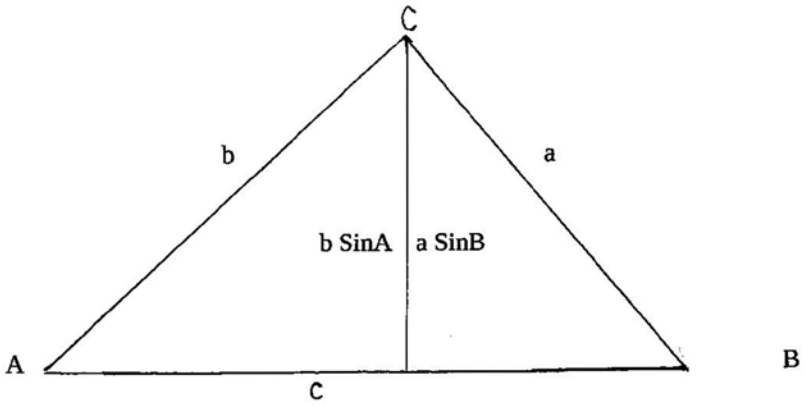
This remarkable formula, "Euler's Formula," unites the fundamental constants of mathematics, namely e, i, π, 1 and 0, in one formula. It also unites the main branches of mathematics: arithmetic (0 and 1), geometry (π) and analysis (e). This topic should not be left without the jolly story of Euler's famous "proof of the existence of God."

11th Grade Appendices

11-1 Law of Sines
11-2 Law of Cosines
11-3 Sine of the Sum of Angles
11-4 Cosine of the Sum of Angles
11-5 Sum and Difference Formulas; Double Angle Formulas
11-6 The Searchlight
11-7 Template for Desarges' Theorems
11-8 Collineation: Imagination and Drawing
11-9 Path Curves
11-10 Geometric Sequences for an Oval
11-11 Projective Pine Cone
11-12 Latitudes and Longitudes of Some Cities

11-1

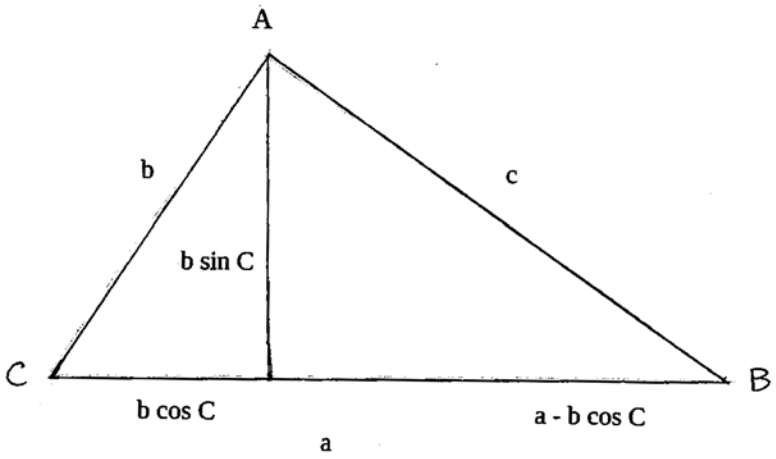
Law of Sines



$$a \sin B = b \sin A$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Law of Cosines



$$c^2 = b^2 \sin^2 C + (a - b \cos C)^2$$

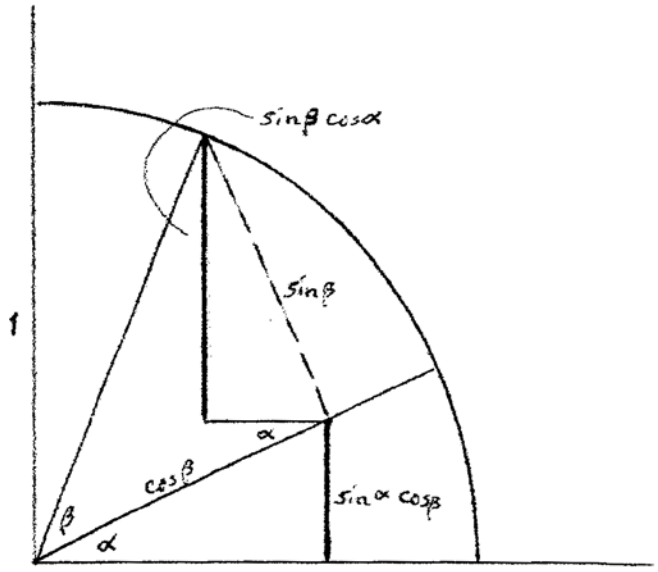
$$= b^2 \sin^2 c + a^2 - 2\,a\,b \cos C + b^2 \cos^2 C$$

$$= a^2 + b^2 (\sin^2 C + \cos^2 C) - 2ab \cos C$$

$$= a^2 + b^2 - 2ab \cos C$$

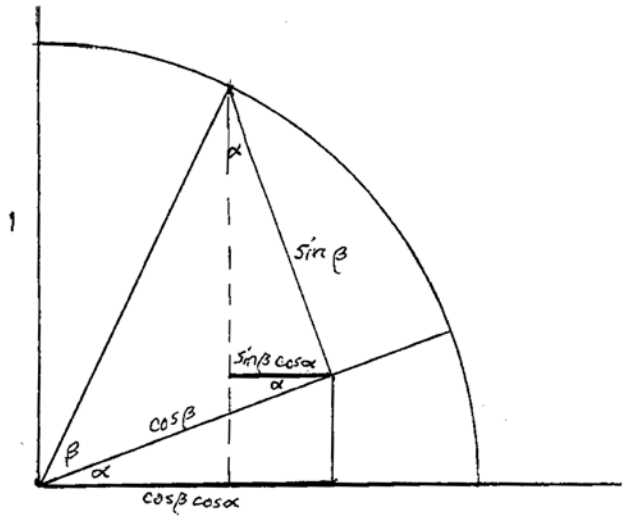
11-3

Sine of the Sum of the Angles



$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

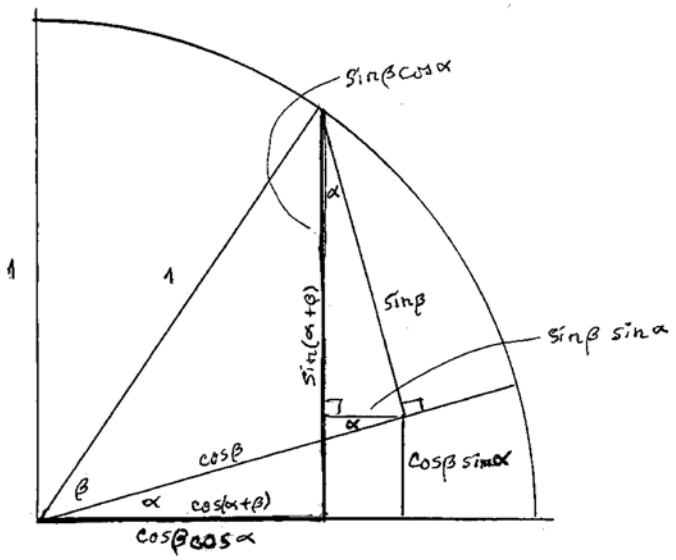
Cosine of the Sum of Angles



$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

11-5

Sum and Difference Formulas; Double Angle Formulas



$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

<u>Difference Formulas</u>

Replace β with -β : $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$

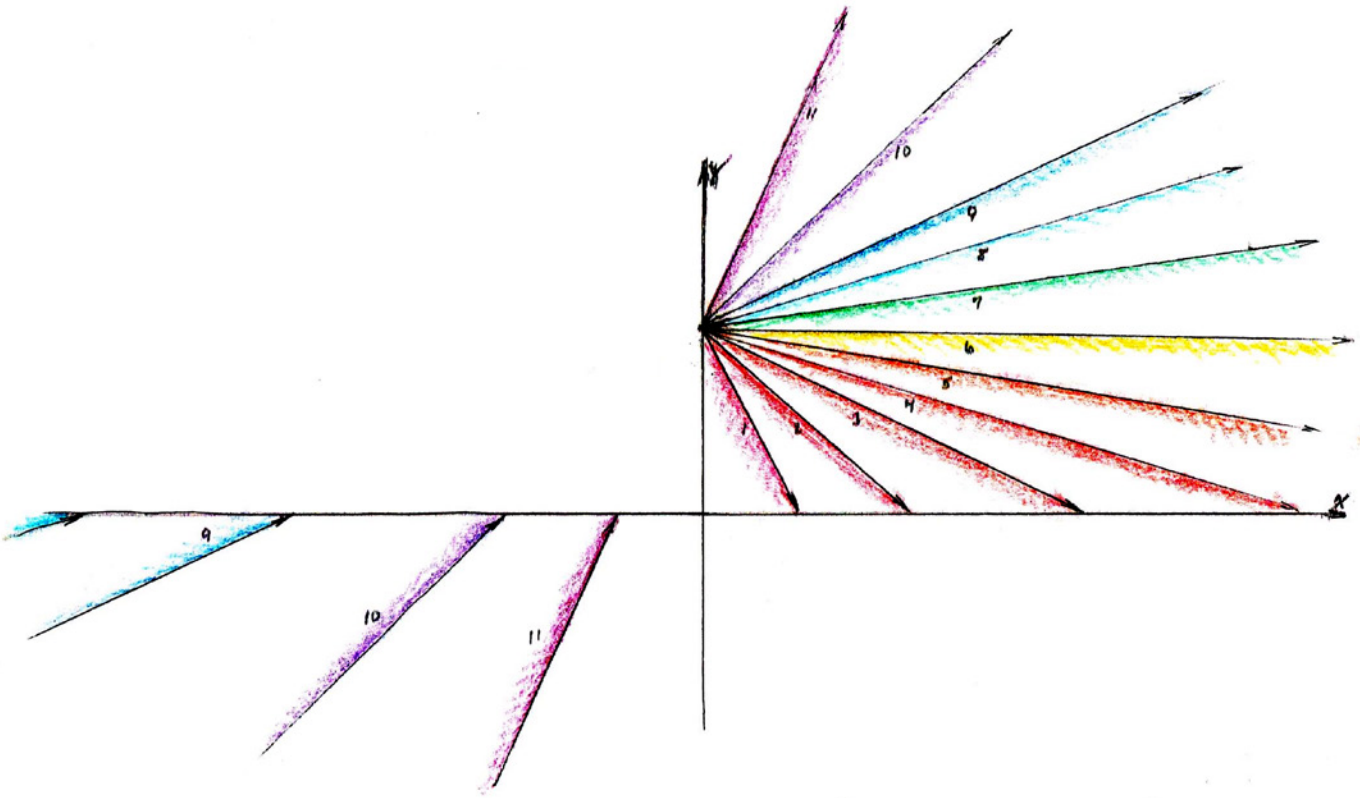
$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

<u>Double Angle Formulas</u>

Replace β with α : $\sin 2\alpha = 2\sin\alpha\cos\alpha$

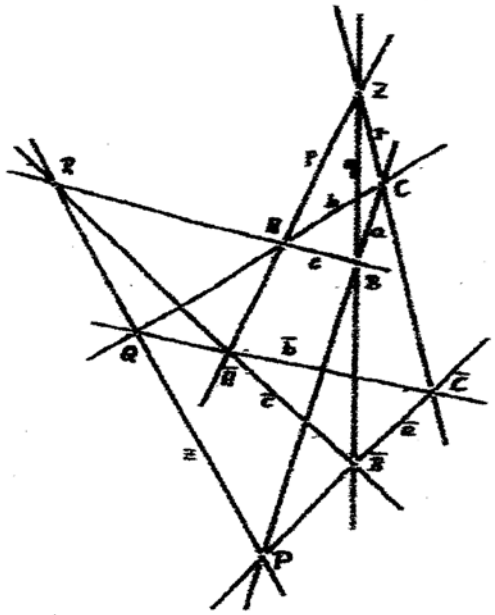
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

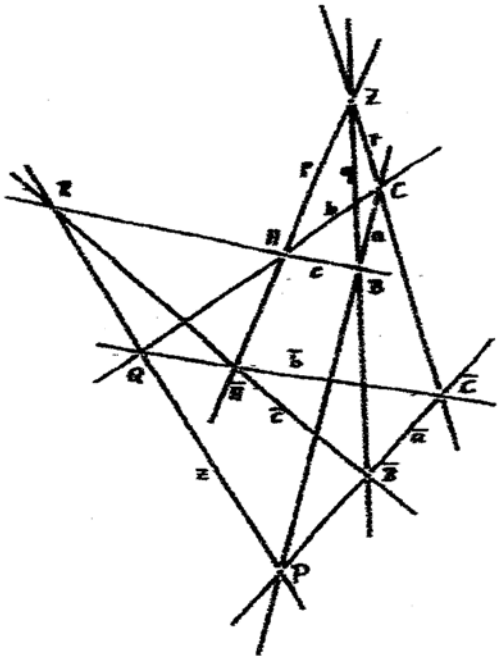
The Searchlight

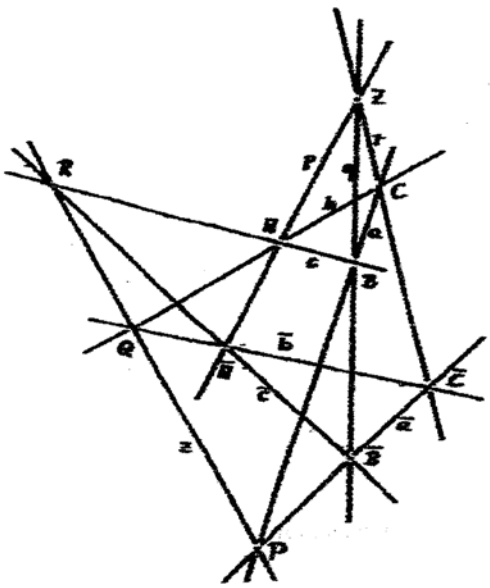


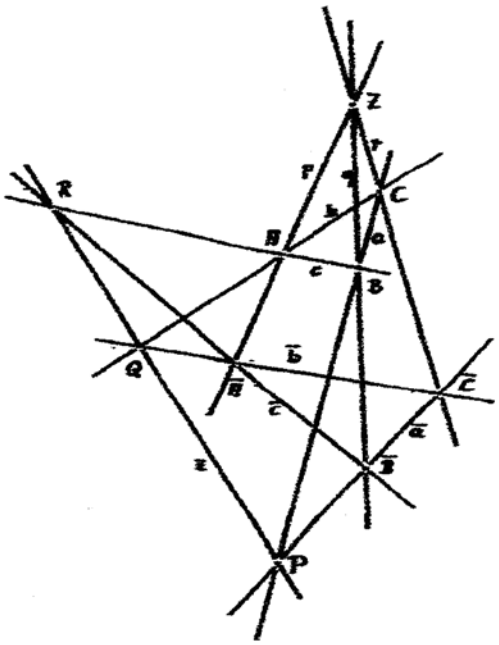
11-7

Template for Desarges' Theorems





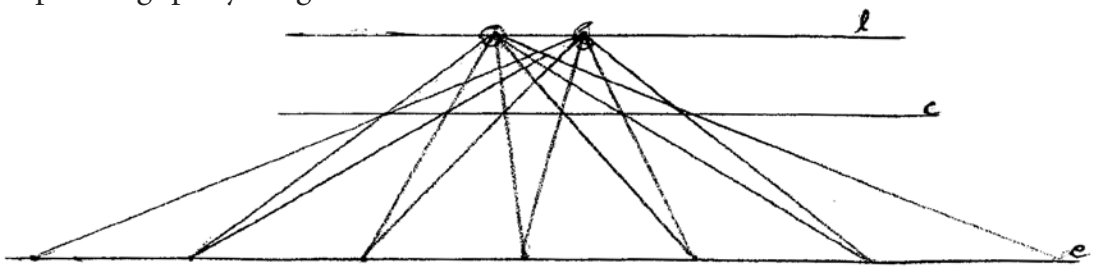




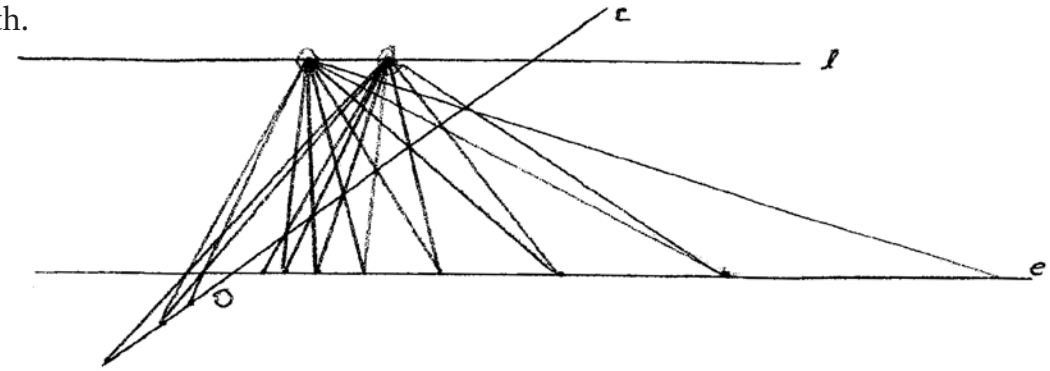
Collineation: Imagination and Drawing

The two Lights (on Line 1) shine down through the Crystal Sphere (line c) onto the Earth (line e), bringing the whole line e into motion. There are three cases: l//c//e, only l//e (or c//e) and none //. Each produces a different "Step measure" or "growth measure" over the Earth.

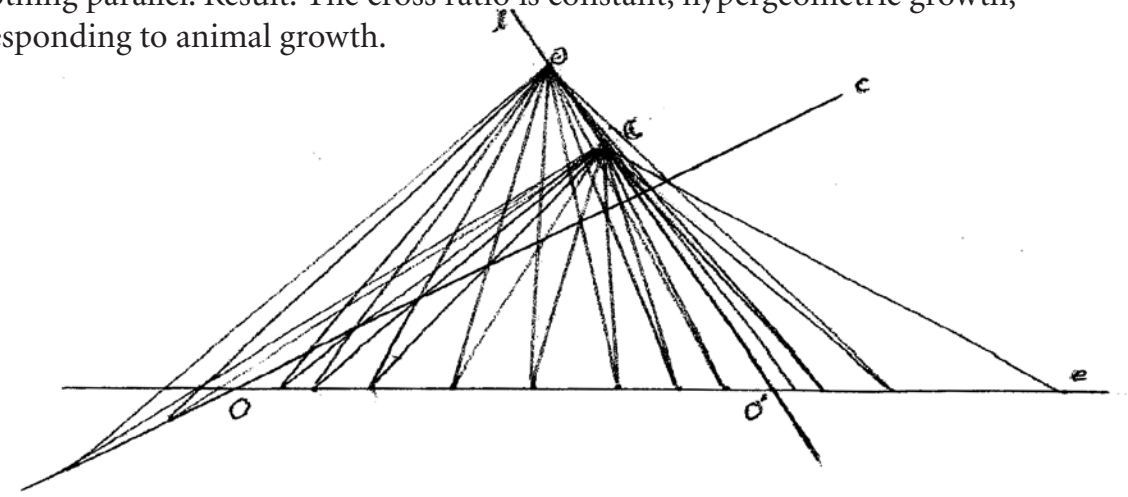
1. l//c//e. Result: Constant difference between steps: arithmetic progression, cooresponding tp crystal growth.



2. l//e. Result: Constant ratio between steps: geometric progression, corresponding to plant growth.

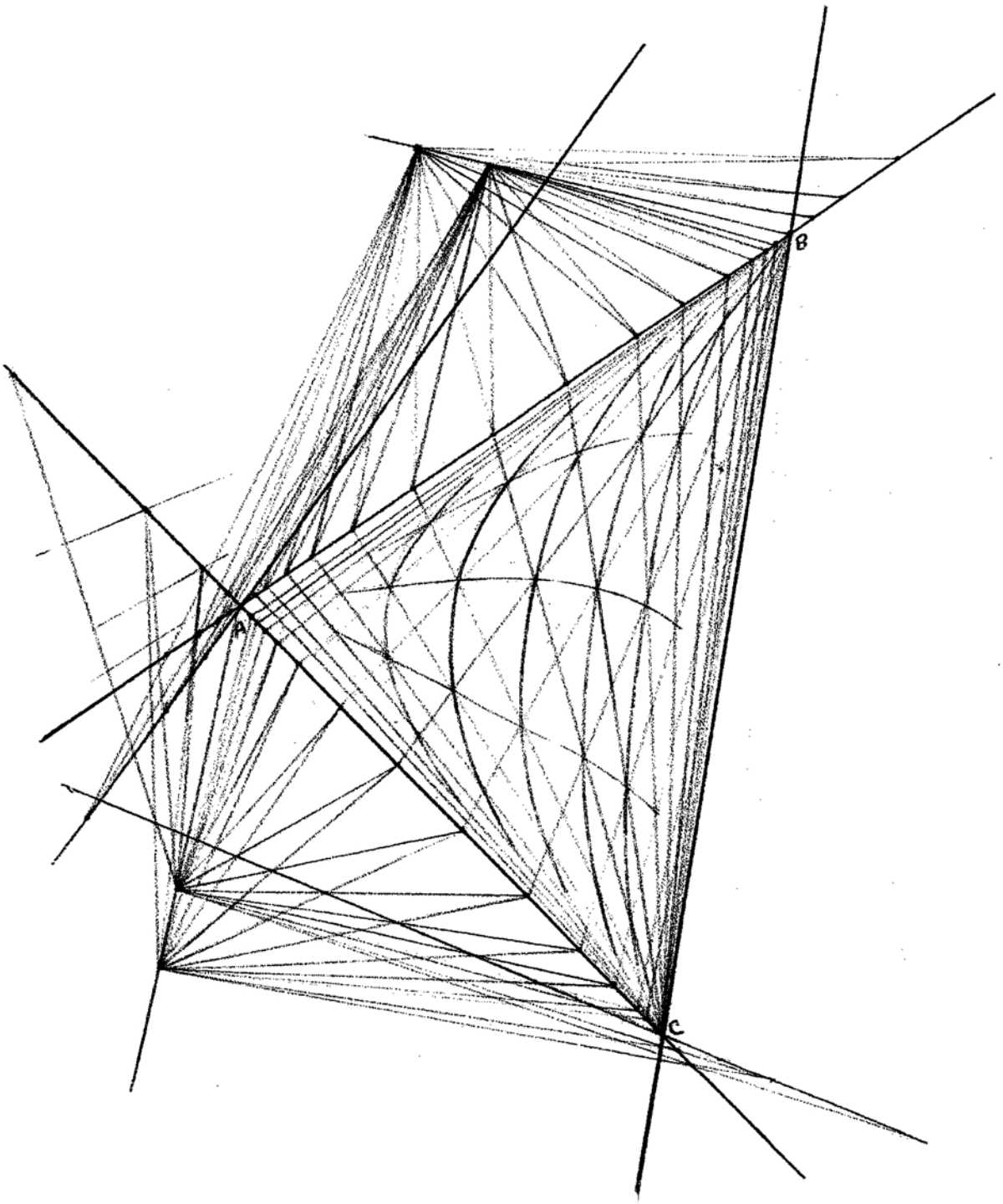


3. Nothing parallel. Result: The cross ratio is constant; hypergeometric growth, corresponding to animal growth.



11-9

Path Curves

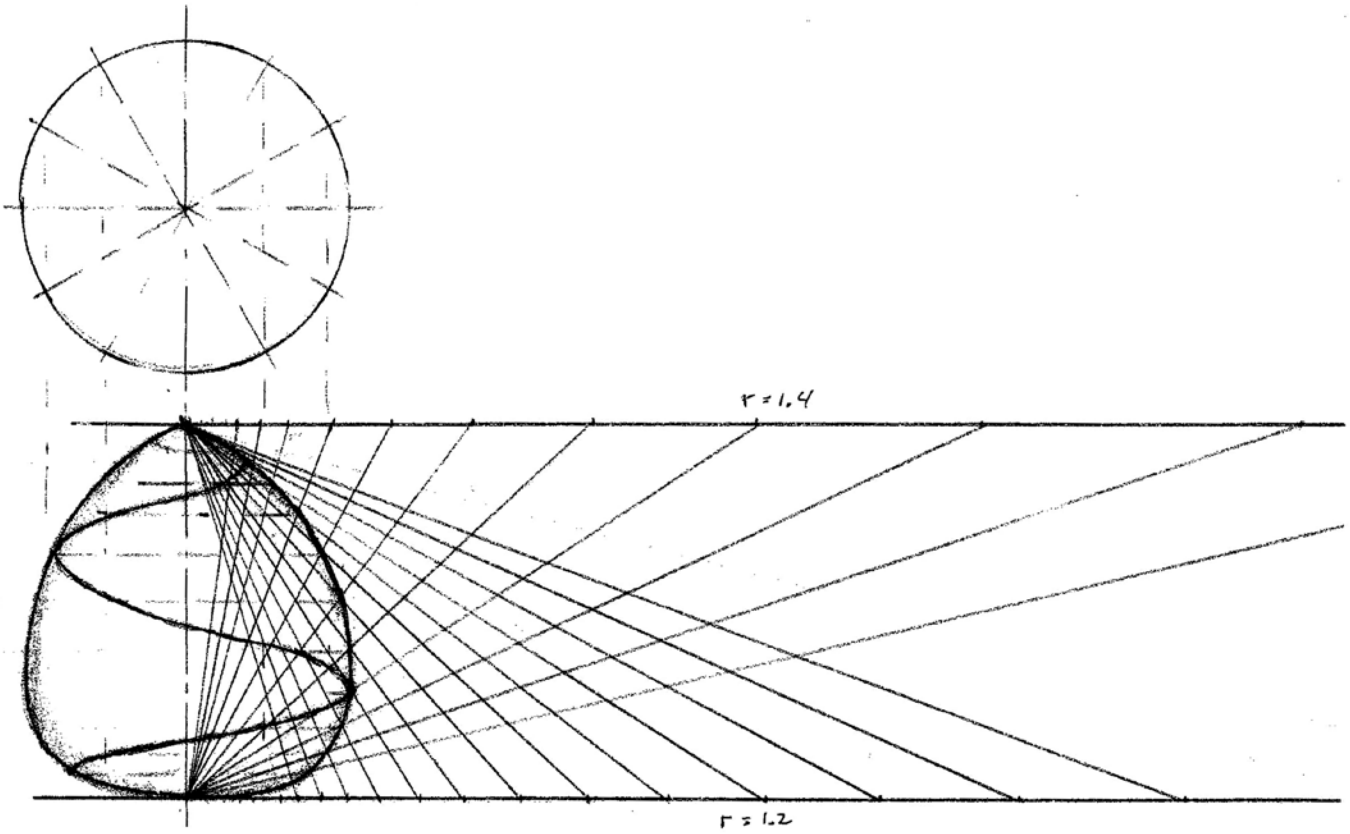


Geometric Sequences for an Oval *(e.g., Pine Cone):*

r = 1.20	r = 1.40
0.13	0.19
0.16	0.26
0.19	0.36
0.23	0.51
0.28	0.74
0.33	1.00
0.40	1.40
0.48	1.96
0.58	2.74
0.69	3.84
0.83	7.53
1.00	10.54
1.20	14.76
1.44	20.66
1.73	28.93
2.07	40.50
2.49	56.69
2.99	
3.58	
4.30	
5.16	
6.19	
7.43	
8.92	
10.70	
12.84	
15.41	
18.49	
22.19	
26.62	
31.95	
38.34	
46.00	
55.21	

11-11

Projective Pine Cone



Latitudes and Longitudes of Some Cities

	City	Country	Latitude	Longitude
8	Accra	Ghana	5.33°N	0.15°W
1	Alice Springs	Australia	23.42°S	133.52°E
7	Bogota	Columbia	4.38°N	74..05°W
5	Cairo	Egypt	30.53°N	31.15°E
3	Cape Cumberland	New Hebrides	14.43°S	166.38°E
	Capetown	South Africa	33.56°S	18.28°E
	Chicago	USA	41.50°N	87.45°W
*	Kampala	Uganda	0.19°N	32.35°E
*	Kisangani	Congo	0.33°N	25.14°E
5	Leningrad	Russia	59.55°N	30..25°E
*	Libreville	Gabon	0.30°N	9.25°E
8	London	England	51.30°N	0.10°W
	Los Angeles	USA	34.00°N	118.15°W
3	Lusaka	Zambia	15.26°S	28.20°E
*	Macapa	Brazil	0.01°N	51.01°W
6	Melbourne	Australia	37.45°S	144.58°E
	Moscow	Russia	55.45°N	37.42°E
7	New York City	USA	40.40°N	73.50°W
4	Oklahoma City	USA	35.28°N	97.33°W
	Paris	France	48.52°N	2.20°E
	Beijing	China	39.50°N	116.25°E
*	Port Victoria	Kenya	0.07°N	34.00°E
2	Prague	Czech Republic	50.05°N	14.25°E
*	Quito	Ecuador	0.14°S	78.30°W
1	Rio de Janiero	Brazil	22.53°S	43.17°W
	Spring Valley, NY	USA	41.07°N	74.02°W
4, 6	Tokyo	Japan	35.40°N	139.45°E
2	Vancouver	Canada	49.13°N	123.06°W

*	City is on the Equator
1, 2, 3, 4	Pairs have the same Latitude
5, 6, 7, 8	Pairs have the same Longitude

Grade 12 Math

Students in the graduating class are ready and able to take on the major issues of our/ their time. Global issues in particular need to be addressed: for example, global warming and poverty (economics block), humans and the animal world (biology), humans and technology (Faust), nuclear power and genetic engineering (chemistry). The area of mathematics which is fundamental to our technological society is calculus. However there are also significant questions which have been raised about the foundations of mathematics itself. New kinds of numbers have appeared, challenging those in and out of the profession.

The main focus of mathematics for the year is calculus. Calculus is fundamental to our world of science and technology, neither of which could thrive without these techniques. The bulk of the time available for mathematics will be devoted to this subject. However there are several other areas which are important. Ideally there would be two mathematics blocks in 12th grade, namely, Calculus and Topics in Modern Mathematics. As a practical matter, it is generally difficult to squeeze even one mathematics block into the 12th grade, given the pressures of other subjects. Thus generally we must make do with the track class.

Calculus

Possible introductory overview: distance, speed and acceleration, their graphical representations and relationships. Background: review Zeno's paradoxes.

A review of the needed algebraic tools is generally needed before plunging in: exponents and radicals, fractions, functions, graphs and slopes. Then students should be ready for differentiation and its rules: powers, exponential and logarithmic functions, trigonometric functions. Integration and its applications follow. Power series and Taylor polynomials offer some surprising relations and a simple proof of Euler's equation. Many other topics could be added as time permits.

There are several calculus texts to choose from. Naturally it would be excellent to show real-world applications as topics are introduced. A serious attempt in this direction is Ron Larson, *Calculus: An Applied Approach*. Although some of the examples appear somewhat artificial, there are some clear indications how calculus could be put to practical use.

An example: The ideal dimensions of a can, if the contents should be maximized and cost minimized a) when all sides use the same material, and b) when the ends cost more, as in soda or beer cans, when they must be flat and resist internal pressure. (See Appendix)

The 12th Grade Math Block

An example of a possible block sequence is described in the Appendix. However if the likelihood of a separate mathematics block is small, some of the topics would be well to add to the track class for the sake of rounding off the mathematical experience and indicating the present state of mathematics.

Mathematics Block

Topics which could be included:

- Set theory
- Cantor and Infinite sets
- Logical basis of mathematics: consistency, completeness, Gödel
- Statistics
- Graph theory
- Voting possibilities and impossibilities
- Linear programming
- Projective geometry of living forms

References

Edwards, Lawrence, *The Vortex of Life*
Kline, Morris *Mathematics: The Loss of Certainty*
Larson, Ron, *Calculus: An Applied Approach*
Miller, Charles D., et al, *Mathematical Ideas*

12th Grade Appendices

12-1 Can Dimensions to Maximize Volume and Minimize Cost
12-2 A 12th Grade Mathematics Block
12-3 The Meaning of Numbers: Some Examples

12-1

Can Dimensions to Maximize Volume and Minimize Cost

1) Soup, tomatoes, etc., with little or no internal pressure.



$S = 2\pi r^2 + 2\pi rh$

$V = \pi r^2 h \quad \text{or} \quad h = \frac{V}{\pi r^2}$

Thus

$$S = 2\pi r^2 + 2\pi r \frac{V}{\pi r^2}$$

$$= 2\pi r^2 + \frac{2V}{r}$$

$$S' = 4\pi r - \frac{2V}{r^2}$$

Set $S' = 0$. Then

$$4\pi r - \frac{2V}{r^2} = 0$$

$$4\pi r = \frac{2V}{r^2}$$

$$r^3 = \frac{2V}{4\pi}$$

$$r = \sqrt[3]{\frac{1}{2\pi}}\sqrt[3]{V}$$

$$= 0.5419\sqrt[3]{V}$$

Suppose V = 12 fluid ounces * 4.8 in³/fluid ounce

$= 57.6 \text{ in}^3$

then $r = 0.5419\sqrt[3]{57.6}$ in

$= 2.1$ in

$h = \frac{V}{\pi r^2}$

$= \frac{57.6 \text{ in}^3}{\pi (2.1)^2}$

$= 4.16''$

Thus the top is about the same distance across (diameter) as the height. This is characteristic of many cans from small to #10.

2) A similar calculation assuming top and bottom material costing twice the sides, as in beer or soda cans, indicates a top whose diameter is half the height, approximately the dimensions of a soda or beer can, which, because of the high internal pressure, must have greater strength (and thus cost) on the top and the bottom.

A 12th Grade Mathematics Block

Twelfth grade students have been studying mathematics since first grade. For many the 12th grade will be their last experience of mathematics. Should they not leave the school with a picture of what mathematics is and where mathematics stands today? The fact is that since the 19th century, professional mathematics has been in disarray; the whole enterprise has been thrown into question, at least in theory. And students should be given a basis for meeting mathematics today in a positive way, making a real connection with mathematics in our time, in the event that they may wish to continue studying mathematics.

The 12th Grade math block may begin with a review of the destruction of the faith in the two pillars of mathematics, namely, algebra and geometry, which took place in the 19th century. (Cf. e.g., Morris Kline, *Mathematics, the Loss of Certainty*)

First of all, non-Euclidean geometries, developed by Gauss, Riemann and Lobachevsky, undermined the faith in an absolute Euclidean geometry and opened the way to the modern understanding of space in Einstein's general relativity theory. (A simple presentation of non-Euclidean geometries and other topics relevant to this block is in Charles D. Miller, et al., *Mathematical Ideas*.) For students who have already studied spherical trigonometry, it should not be difficult to understand how these geometries can arise.

Next comes the somewhat more abstruse shaking of the foundations of algebra, which began with the struggles of Hamilton to generalize complex numbers. The tremendous shock to Hamilton and other mathematicians when his "quaternions" violated the "commutative" axiom may be appreciated by those who have taken for granted for years that 2x3 = 3x2. Multiplication of vectors can be used to illustrate non-commutative groups, also multiplication of matrices. The examples would depend on what had been part of the curriculum in 11th grade, but usually vectors are studied in physics and an introduction to matrices is a common topic in mathematics.

How can it be that some basic rules of mathematics, seemingly obvious, are not universally valid? What are the foundations of Mathematics? Can it be built up logically from self-evident first principles? There have been three main attempts to do this in the 20th century: attempts to answer the "Foundation Question." To understand the first attempt and its failure, we must understand Cantor's theory of infinite numbers.

Cantor developed his theory of infinite sets between 1874 and 1884. While the ancients had a "horror of a vacuum," modern man takes a vacuum in stride. Yet even the great 19th century mathematician Gauss described the "horror of the actual infinite." What did he mean? Gauss could certainly speak about infinity as "potential," something you could approach. But he rebelled at the idea of actually grasping this idea and speaking about it as a reality, as a number, such as the natural numbers, or even vectors or matrices. However this is precisely what Cantor did. He even developed the idea of comparing the size of these numbers and, further, proposed a hierarchy of infinite numbers. Gauss and most other mathematicians rejected Cantor's ideas. Cantor gradually went insane. Was this a result of his rejection or of his wrestling with the infinite?

[*Note for Waldorf teachers:* At the time of the end of the "dark age" (Kali Yuga), Cantor introduced his theory of a hierarchy of infinities, of comparing infinite numbers. If we recognize infinity as our picture in the finite realm of the infinite Godhead, Cantor has provided us with a picture of the hierarchies of spiritual beings.]

Out of Cantor's theory of trans-finite numbers comes the paradox which defeats the first attempt to found mathematics firmly. Bertrand Russell and Alfred North Whitehead proposed to found mathematics solidly on the rock of pure logic, that is, on the theory of sets. However set theory produces the following conundrum: Let us take the subsets of the largest set. But this number, the number of subsets, is larger than the original "largest" set, producing a logical contradiction. As Russell admitted in 1959, "The splendid certainty which I had always hoped to find in mathematics was lost in a bewildering maze."

A simpler example of a self-referent statement is, "I am lying." The statement says that the speaker is lying. But if that is a lie, then the speaker is not lying.

Already in classical Greek times such paradoxes were recognized. Eubulides, 4th century BC: "This statement I am now making is false." Epimenides, 6th century BC, Cretan philosopher: "Cretans are always liars." (cf. Howard Eaves, *An Introduction to the History of Mathematics*)

Questions for class: You are a traveler in a land where everyone lies. You come to a fork in the road. What should you ask a bystander in order to find the correct route to your destination? And a challenge would be: You are a traveler in a land where some people lie and some tell the truth. What one question can you ask the native when you come to the fork in the road? [*Students should discover:* "If I were to ask you which is the correct road, what would you answer?"]

These "self-referent" statements tolled the death knell for the faith in pure logic. [*Note to teacher:* This failure came at the dawn of ego-consciousness, the beginning of the capacity to see oneself more objectively, to begin thinking about thinking itself.]

The second program to refound mathematics on firm ground is especially associated with David Hilbert. He sought a more carefully-formulated axiomatic system (cf., e.g., Eves). However this quest came to an abrupt halt in the early 1930s with a series of papers by Kurt Gödel. Gödel proved that any mathematical system large enough to include the arithmetic of whole numbers (and these are certainly the simplest numbers of all) could not be proven to be consistent.

[*Note to teacher:* It seems singularly appropriate that this proof of man's incapacity to form a self-contained consistent system of thought came at the time of the re-appearance of Christ in the etheric and of the gradual awakening of the capacity to once again perceive beyond the physical world.]

The third program, by the Intuitionist School, sought to build mathematics up simply on the basis of the natural numbers and human intuition. However the higher faculties of reason and logic still had to come into play. The program was never able to arrive at very many important mathematical results.

Such is the state of mathematics today: uncertainty about its foundation and its basis in reality. Yet the power of mathematics is undisputed, from rockets, construction, finance, design. Is there another way to approach mathematics? We could of course simply resort to Pragmatism: Whatever works is True. This may be sufficient philosophy for an engineer, but surely it is unsatisfying.

Amos suggested building up observations of natural phenomena: pine cones, sunflowers, human aesthetics, the Golden Section. These numbers arising naturally in nature, suggest that there may be something real about mathematics after all. Many other examples could be assembled. Even if there is no completely logical foundation of mathematics, still mathematical reality itself is real in our minds and in nature. Mathematical entities are objectively real in a mental (spiritual) realm. (neo-Platonism)

What about numbers themselves? An appreciation of the individuality and inner nature of numbers comes with the presentation of some of the results of number theory.

The block may conclude with some projective geometric drawing, showing how intimately mathematics is connected to living forms. (See Edwards, *The Vortex of Life*) The forms of pine cones and buds arise from simple projective transformations (collineations), demonstrating the objective power and reality of mathematics.

Other possible topics: topology, statistics. linear programming...

12-3

The Meaning of Numbers: Some Examples

Pythagoras: Numbers originate in the Supreme One, God, source of universal harmony.
Thus the largest number is One.
Numbers are divine faculties in action.
Goethe's Faust: Numbers are from the realm of the Mothers.

Special Numbers (Pythagoras):

One. God, unity, indivisible, unchangeable, creator, father
Fire, male principles
Manifests as:

Two. Union of active and passive, masculine and femine, God and Nature, earthly and heavenly natures

Three. Real world: Man as threefold being: body, soul and spirit; nerve-sense, rhythmic and metabolic-limb systems

Four. Pythagoras: the Sacred Tetrad because God is revealed four ways:
1. Instinctively we see myriad in nature, thus polytheism
2. Reason sees duality: good vs evil, thus Zoroaster
3. Thought sees a threefold God, in both Christianity and Hinduism
4. God is a unity as the universal unity of the world, therefore monotheism.

Pythagorean saying: "I swear by the One who is engraved in our hearts: The sacred Tetrad, mighty and pure Symbol, source of Nature, archetype of the gods."

Pythagoras: All numbers derive from the first four.

Some other special numbers: 7, 12, 30, 360



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